Generalized Calogero–Moser–Sutherland systems

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math-ph/0503020

City University, 22.04.2005

Calogero–Moser–Sutherland operator

$$L = \Delta - \sum_{i<j}^{n} \frac{2m(m+1)}{\sinh^2(x_i - x_j)}$$

Quantum complete integrability:

$$L_1 = \partial_1 + \ldots + \partial_n,$$

$$L_2 = L,$$

$$L_s = \partial_1^s + \ldots + \partial_n^s + \text{ lower order terms},$$

$s = 1, 2, \ldots$.

$$[L_s, L_{\tilde{s}}] = 0.$$

$L_1, \ldots, L_n$ are algebraically independent.

If $m \in \mathbb{Z}_+$ then there are additional quantum integrals (Chalykh,Veselov'90). For example,

$$M_t = \prod_{i<j}^{n} (\partial_i - \partial_j)^{2m} \partial_t + \text{ lower order terms},$$

$1 \leqslant t \leqslant n$.

$$[M_t, M_{\tilde{t}}] = 0,\ [M_t, L_s] = 0.$$

Generalized CMS systems

Consider configuration $\mathcal{A} = (A, m)$ where A is a finite set of non-collinear vectors α, $\alpha \in \mathbb{C}^n$; m is a multiplicity function, $m(\alpha) = m_\alpha \in \mathbb{Z}_+$.

$$L^{\mathcal{A}} = \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{\sinh^2(\alpha, x)}$$

Examples of configurations $\mathcal{A}$ leading to supercomplete rings of quantum integrals containing $L^{\mathcal{A}}$.

1. Calogero–Moser–Sutherland case, $\boldsymbol{A = A_{n-1}}$.

$$\alpha = \alpha_{ij} = e_i - e_j, \quad 1 \leqslant i < j \leqslant n, \quad m_{\alpha_{ij}} = m$$

2. $\boldsymbol{A = C_n}$

$$e_i, \quad 1 \leqslant i \leqslant n, \quad m_i = l$$

$$\frac{1}{2}(e_i \pm e_j), \quad 1 \leqslant i < j \leqslant n, \quad m_{ij} = m$$

3. $\boldsymbol{A = R}$ – root system [Olshanetsky, Perelomov, 1977]

The set of hyperplanes $\{(\alpha, x) = 0 | \alpha \in A\}$ is invariant under reflections around them. The multiplicity function $m = m(\alpha)$ is also invariant under Weyl group.

4. $\boldsymbol{\mathcal{A}_{n,1}(m)}$ [Chalykh, F., Veselov, 1996]

$$e_i - e_j \quad 1 \leqslant i < j \leqslant n \quad m_{ij} = m$$

$$e_i - \sqrt{m} e_{n+1} \quad 1 \leqslant i \leqslant n \quad m_{i,n+1} = 1$$

$\mathcal{A}_{n,1}(1)$ is root system A_n with multiplicity 1

5. $\boldsymbol{\mathcal{C}_n(l, m)}$ [Chalykh, F., Veselov, 1999]

$$\bar{e}_i = \sqrt{2m+1} e_i, \quad 1 \leqslant i \leqslant n-1, \quad m_i = l$$

$$\bar{e}_n = \sqrt{2l+1}\, e_n, \qquad m_n = m,$$

$$\frac{\bar{e}_i \pm \bar{e}_j}{2}, \quad 1 \leqslant i < j \leqslant n-1, \quad m_{ij} = \frac{2l+1}{2m+1}$$

$$\frac{\bar{e}_i \pm \bar{e}_n}{2}, \quad 1 \leqslant i \leqslant n-1, \quad m_{in} = 1.$$

We assume $\frac{2l+1}{2m+1} \in \mathbb{Z}_+$; $l, m \in \mathbb{Z}_+$.

$\mathcal{C}_n(m,m)$ is root system C_n with multiplicities $m, 1$

6. $\mathcal{A}_{n,2}(m)$ [Chalykh, Veselov, 2001]

$$e_i - e_j \quad 1 \leqslant i < j \leqslant n-1 \quad m_{ij} = m$$

$$e_i - \sqrt{m} e_n \quad 1 \leqslant i \leqslant n \quad m_{i,n} = 1$$

$$\sqrt{-m-1} e_0 - e_i \quad 1 \leqslant i \leqslant n \quad m_{o,i} = 1$$

$$\sqrt{-m-1} e_0 - \sqrt{m} e_n \qquad m_{0,n} = 1$$

Baker–Akhiezer functions for generalized Calogero–Moser–Sutherland systems

[Chalykh, Veselov, 1990; root systems]

For $\mathcal{A} = (A, m) = \mathcal{R}, \mathcal{A}_{n,1}(m), \mathcal{C}_n(l, m), \mathcal{A}_{n,2}(m)$ there exist Baker–Akhiezer functions

$$\psi^{\mathcal{A}}(k, x) = P^{\mathcal{A}}(k, x)e^{(k,x)},$$

$k, x \in \mathbb{C}^n$, $(k, x) = k_1x_1 + \ldots + k_nx_n$, where polynomials $P^{\mathcal{A}}$ have the form

$P^{\mathcal{A}}(k, x) = \prod_{\alpha \in A} (k, \alpha)^{m_\alpha} +$ lower order terms in k,

such that the following conditions hold.

For any vector $\alpha \in A$

$$\psi^{\mathcal{A}}(k + s\alpha) \equiv \psi^{\mathcal{A}}(k - s\alpha)$$

at $(k, \alpha) = 0$ for $s = 1, \ldots, m_\alpha$ for $\mathcal{A} = \mathcal{R}, \mathcal{A}_{n,1}(m)$, $\mathcal{C}_n(l, m)$ (and more complicated for $\mathcal{A} = \mathcal{A}_{n,2}(m)$).

Baker–Akhiezer function $\psi^{\mathcal{A}}(k,x)$ satisfies

$$\Big(\Delta - \sum_{\alpha\in A} \frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{\sinh^2(\alpha,x)}\Big)\psi^{\mathcal{A}} = k^2\psi^{\mathcal{A}}.$$

Define **ring $\boldsymbol{R}^{\mathcal{A}}$** consisting of polynomials $p(k)$ such that

$$p(k+s\alpha) \equiv p(k-s\alpha) \qquad \text{at } (k,\alpha)=0$$

for $s = 1,\dots,m_\alpha$ for any $\alpha \in A$.

Theorem [Chalykh, Veselov, 1990]

For any $p(k) \in R^{\mathcal{A}}$ there exists differential operator

$$L_p(x,\partial_x) = p(\partial_x) + \text{lower order terms}$$

such that

$$L_p(x,\partial_x)\psi^{\mathcal{A}} = p(k)\psi^{\mathcal{A}},$$

and $[L_{p_1}, L_{p_2}] = 0$ for any $p_1, p_2 \in R^{\mathcal{A}}$.

Bispectrality. Dual difference operators.

$\mathcal{A} = (A_n, m)$ – **Ruijsenaars operator** (1987)

$$D^{\mathcal{A}_n} = \sum_{i=1}^{n+1} \prod_{j \neq i}^{n+1} \Big(1 - \frac{2m}{k_i - k_j}\Big) T^i$$

where

$T_i f(k_1, \dots, k_i, \dots, k_{n+1}) = f(k_1, \dots, k_i + 2, \dots, k_{n+1})$.

Then $D^{\mathcal{A}_n} \psi^{\mathcal{A}_n} = (e^{2x_1} + \dots + e^{2x_{n+1}}) \psi^{\mathcal{A}_n}$

[Chalykh,2000]

$\mathcal{A} = \mathcal{A}_{n,1}(m)$

$$D^{\mathcal{A}_{n,1}(m)} = a_1 T_1 + \dots + a_n T_n + a_{n+1} T_{n+1}^{\sqrt{m}},$$

where for $i = 1, \ldots, n$

$$a_i = \left(1 - \frac{2}{k_i - \sqrt{m}k_{n+1} + 1 - m}\right) \prod_{j \neq i}^{n} \left(1 - \frac{2m}{k_i - k_j}\right)$$

and

$$a_{n+1} = \frac{1}{m} \prod_{i=1}^{n} \left(1 + \frac{2m}{k_i - \sqrt{m}k_{n+1} + 1 - m}\right),$$

$$T_{n+1}^{\sqrt{m}} f(k_1, \ldots, k_{n+1}) = f(k_1, \ldots, k_{n+1} + 2\sqrt{m}).$$

Theorem [Chalykh, 2000]

$$D^{\mathcal{A}_{n,1}(m)} \psi^{\mathcal{A}_{n,1}(m)} =$$

$$(e^{2x_1} + \ldots + e^{2x_n} + \frac{1}{m} e^{2\sqrt{m}x_{n+1}}) \psi^{\mathcal{A}_{n,1}(m)}$$

Macdonald operator (1988)

$\mathcal{A} = \mathcal{R}$ − root system of type $A_n, B_n, C_n, D_n, E_6, E_7$.
π − minuscule coweight for $\frac{1}{2}R^\vee$: $(\pi, \alpha) \in \{0, -1, 1\}$ for all $\alpha \in \frac{1}{2}R^\vee$.

$$D_\pi^{\mathcal{R}} = \sum_{\substack{\tau = w\pi \\ w \in W}} \left(\prod_{\substack{\alpha \in \frac{1}{2}(R^\vee \cup (-R^\vee)) \\ (\alpha,\tau)=1}} \left(1 - \frac{m_\alpha}{(\alpha, k)}\right) \right) T^\tau,$$

where W is the corresponding Weyl group, and $T^\tau f(k) = f(k+\tau)$.

Theorem [Chalykh, 2000]

$$D_\pi^{\mathcal{R}} \psi^{\mathcal{R}} = \Big(\sum_{w \in W} e^{(w\pi, x)} \Big) \psi^{\mathcal{R}}$$

Dual operator for $\mathcal{C}_n(l,m)$

$$\bar{e}_i = \sqrt{2m+1}e_i, \quad 1 \leqslant i \leqslant n-1, \quad m_i = l$$

$$\bar{e}_n = \sqrt{2l+1}\,e_n, \qquad m_n = m,$$

$$\frac{\bar{e}_i \pm \bar{e}_j}{2}, \quad 1 \leqslant i < j \leqslant n-1, \quad m_{ij} = \frac{2l+1}{2m+1}$$

$$\frac{\bar{e}_i \pm \bar{e}_n}{2}, \quad 1 \leqslant i \leqslant n-1, \quad m_{in} = 1.$$

Define

$$D^{\mathcal{C}_n(l,m)} = \sum_{i=1}^{n} a_i^+ T_i^+ + a_i^- T_i^-$$

where for $1 \leqslant i \leqslant n-1$

$$T_i^{\pm} f(k_1, \ldots, k_n) = f(k_1, \ldots, k_i \pm \sqrt{2m+1}, \ldots, k_n),$$

$$T_n^{\pm} f(k_1, \ldots, k_n) = f(k_1, \ldots, k_n \pm \sqrt{2l+1}).$$

Then coefficients $a_i^{\pm}$ are defined by

$$a_i^{\pm} = \prod_{j=1}^{n} a_{ij}^{\pm}, \qquad i = 1, \ldots, n,$$

where for $1 \leqslant i, j \leqslant n-1$, $i \neq j$

$$a_{ij}^{\pm} = \left(1 - \frac{2l+1}{\pm\bar{k}_i + \bar{k}_j}\right)\left(1 - \frac{2l+1}{\pm\bar{k}_i - \bar{k}_j}\right),$$

$$a_{ii}^{\pm} = \frac{1}{2m+1}\left(1 - \frac{(2m+1)l}{\pm\bar{k}_i}\right),$$

$$a_{in}^{\pm} = \left(1 - \frac{2m+1}{\pm\bar{k}_i + \bar{k}_n - l + m}\right)\left(1 - \frac{2m+1}{\pm\bar{k}_i - \bar{k}_n - l + m}\right),$$

$$a_{nj}^{\pm} = \left(1 - \frac{2l+1}{\pm\bar{k}_n + \bar{k}_j + l - m}\right)\left(1 - \frac{2l+1}{\pm\bar{k}_n - \bar{k}_j + l - m}\right),$$

$$a_{nn}^{\pm} = \frac{1}{2l+1}\left(1 - \frac{(2l+1)m}{\pm\bar{k}_n}\right),$$

where we use notation $\bar{k}_i = \sqrt{2m+1}\, k_i$ for $i = 1, \ldots, n-1$, and $\bar{k}_n = \sqrt{2l+1}\, k_n$.

Theorem [F]

$$D^{\mathcal{C}_n(l,m)} \psi^{\mathcal{C}_n(l,m)} =$$

$$\left(\frac{2}{2m+1}\sum_{j=1}^{n-1}\cosh\bar{x}_j+\frac{2}{2l+1}\cosh\bar{x}_n\right)\psi^{\mathcal{C}_n(l,m)},$$

where $\bar{x}_j=\sqrt{2m+1}x_j$, $\bar{x}_n=\sqrt{2l+1}x_n$.

Dual operator for $\mathcal{A}_{n,2}(m)$

$$e_i-e_j,\quad 1\leqslant i<j\leqslant n-1,\quad m_{ij}=m$$

$$\bar{e}_0-e_i, e_i-\bar{e}_n,\ \bar{e}_0-\bar{e}_n,$$

where

$$\bar{e}_0=\sqrt{-m-1}e_0,\ \bar{e}_n=\sqrt{m}e_n,$$

$1\leqslant i\leqslant n-1$, and $m_{in}=m_{0i}=m_{0n}=1$.

Define

$$D=-\frac{1}{m+1}\left(1+\frac{2(m+1)}{\bar{k}_0-\bar{k}_n-2m-1}\right)\times$$

$$\prod_{j=1}^{n-1}\left(1+\frac{2(m+1)}{\bar{k}_0-k_j-m-2}\right)T_0+$$

$$\sum_{i=1}^{n-1} \left(1 - \frac{2}{k_i - \bar{k}_0 + m + 2}\right) \left(1 - \frac{2}{k_i - \bar{k}_n - m + 1}\right) \times$$

$$\prod_{j=1}^{n-1} \left(1 - \frac{2m}{k_i - k_j}\right) T_i +$$

$$+ \frac{1}{m} \left(1 - \frac{2m}{\bar{k}_n - \bar{k}_0 + 2m + 1}\right) \times$$

$$\prod_{j=1}^{n-1} \left(1 - \frac{2m}{\bar{k}_n - k_j + m - 1}\right) T_n.$$

Theorem [F]

$$D^{\mathcal{A}_{n,2}(m)} \psi^{\mathcal{A}_{n,2}(m)} =$$

$$\Big(\sum_{i=0}^{n} \frac{1}{(\bar{e}_i, \bar{e}_i)} e^{2\bar{x}_i}\Big) \psi^{\mathcal{A}_{n,2}(m)},$$

where $\bar{e}_i = e_i$ for $1 \leqslant i \leqslant n-1$.

Construction of BA functions

Chalykh, 2000: $\mathcal{A} = \mathcal{R}, \mathcal{A}_{n,1}(m)$.

Theorem

For $\mathcal{A} = \mathcal{R}, \mathcal{A}_{n,1}(m), \mathcal{A}_{n,2}(m), \mathcal{C}_n(l,m)$,

$$\psi^{\mathcal{A}} = c^{\mathcal{A}}(x)\Big(D^{\mathcal{A}} - \lambda^{\mathcal{A}}(x)\Big)^M Q(k) e^{(k,x)},$$

where

$$Q(k) = \prod_{\alpha \in A} \prod_{i=1}^{m_\alpha} (k + i\alpha, \alpha)(k - i\alpha, \alpha),$$

$\lambda^{\mathcal{A}}(x) = (\psi^{\mathcal{A}})^{-1} D^{\mathcal{A}}(\psi^{\mathcal{A}})$, $M = \sum_{\alpha \in A} m_\alpha$.

Commuting difference operators

Let $\mathcal{A}$ be one of $R, \mathcal{A}_{n,1}(m), \mathcal{A}_{n,2}(m), \mathcal{C}_n(l,m)$, let $p \in R^{\mathcal{A}}$. Define

$$D_p = ad_{D^{\mathcal{A}}}^{degp} p(k),$$

where $ad_A B = A \circ B - B \circ A$.

Theorem

For any $p_1, p_2 \in R^{\mathcal{A}}$

$$[D_{p_1}, D_{p_2}] = 0,$$

and also $[D^{\mathcal{A}}, D_{p_1}] = 0$. These operators satisfy

$$D_{p_1} \psi^{\mathcal{A}} = \widehat{p}_1(x) \psi^{\mathcal{A}}.$$

Integrability at generic coupling constants

$\mathcal{R}$: Take p_s - basic invariants

$\mathcal{A}_{n,1}(m)$:

$p_s = k_1^s + \ldots + k_{n-1}^s + m^{\frac{s-2}{2}} k_n^s +$lower order terms

$\mathcal{C}_n(l,m)$:

$$p_s = k_1^{2s} + \ldots + k_{n-1}^{2s} + (\frac{2l+1}{2m+1})^{s-1} k_n^{2s} + \ldots$$

$\mathcal{A}_{n,2}(m)$:

$p_s = (-1-m)^{\frac{s-2}{2}} k_0^s + k_1^s + \ldots + k_{n-1}^s + m^{\frac{s-2}{2}} k_n^s + \ldots$

Theorem [Chalykh, F., Veselov'99-05] In every case there exist differential operators

$$L_s = p_s(\partial_x) + \text{lower order terms}$$

such that $[L_s, L_{\tilde{s}}] = 0$, and $L_2 = L^{\mathcal{A}}$ (or $L_1 = L^{\mathcal{A}}$). The difference operators $D_s = ad_{D^{\mathcal{A}}}^{deg p_s} p_s$ also commute $[D_s, D_{\tilde{s}}] = 0$.

Restrictions for configurations admitting BA functions

Let $\mathcal{A} = (A, m)$ admit BA function. Let $A_+ \subset (A \cup (-A))$ be a positive subsystem with an edge vector α.

Theorem [F]

$$A_+ \backslash \alpha = B_1 \sqcup \ldots \sqcup B_N$$

such that for all l, $1 \leqslant l \leqslant N$

1) for any $\beta, \gamma \in B_l$

$$\beta - \gamma = n_{\beta\gamma}\alpha,$$

with $n_{\beta\gamma} \in \mathbb{Z}$;

2) $\sum_{\beta \in B_l} m_\beta (m_\beta + 1)(\beta, \beta)(\alpha, \beta)^{2s-1} = 0$, where $1 \leqslant s \leqslant m_\alpha$.

Structure of $\mathcal{A}$ in general