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Graphic deviation

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ABSTRACT

Given a sequence of n nonnegative integers how can we find the graphs which achieve the minimal deviation from that sequence? This extends the classical problem regarding what sequences are "graphic", that is, can be the degrees of a simple graph, to issues regarding arbitrary sequences. In this context, we investigate properties of the "minimal graphs". We shall demonstrate how a variation on the Havel–Hakimi algorithm can supply the value of the minimal possible deviation, and how consideration of the Ruch–Gutman condition and the Ferrer diagram can yield the complete set of graphs achieving this minimum. An application of this analysis is to a population of individuals represented by vertices, interactions between pairs by edges and in which each individual has a preferred range for their number of links to other individuals. Individuals adjust their links according to their preferred range and the graph evolves towards some set of graphs which achieve the minimal possible deviation. This Markov chain is defined but detailed analysis is omitted.

1. Introduction

For $n \in \mathbb{N}$ ($\mathbb{N}$ being the set of nonnegative integers), we consider the set $\mathbf{S_n}$ of n-element sequences $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ where $(n-1) \geq u_1 \geq u_2 \ldots u_i \geq u_{i+1} \ldots u_n \geq 0$. There is a considerable body of work [2–6] addressing the question of which elements of $\mathbf{S_n}$ are graphic, that is for which $\mathbf{u} \in \mathbf{S_n}$ does there exist a simple graph whose vertices have precisely the degrees (number of neighbours) u_i; there may exist more than one. We shall denote the set of graphic sequences by $\mathbf{H_n}$. However, there is, to our knowledge, nothing addressing the question of "how far from graphic" a member of $\mathbf{S_n}$ is.

Suppose we have a set of n individuals labelled $\{1, 2, \ldots, n\}$, and that associated with vertex i is a pair of integers $0 \leq m_i \leq M_i \leq (n-1)$. We investigate whether there are simple graphs on these n vertices with degrees u_i such that $m_i \leq u_i \leq M_i$, and more generally what graphs are closest to achieving these inequalities in a sense defined below. This investigation is facilitated by introducing a process on the possible graphs which we index with t, and refer to t as time. Thus suppose at time t there exists a graph with n vertices, $\mathbf{X_t}$. We denote a state of the system by the $n \times n$ adjacency matrix $X = (x_{ij})$ (thus $x_{ii} = 0, x_{ij} = 1$ if there is an edge (i, j) and is 0 otherwise). We write $\mathbf{X_t} = \mathbf{x}$ when $\mathbf{X_t}$ is a graph with adjacency matrix $\mathbf{x}$. The sequence associated with the graph $\mathbf{x}$ is $\mathbf{u}$, where the ith element of $\mathbf{u}$, u_i, is the degree of the ith vertex of $\mathbf{x}$. If $m_i \leq u_i \leq M_i$ we say that vertex i is *Neutral*, in which case no action is taken, if $u_i < m_i$ we say vertex i is a *Joiner* and then, if possible, we select a vertex j to which it is not currently linked and add the edge (i, j) to it, while if $u_i > M_i$ then i is said to be a *Breaker* and then, if possible, we select a vertex j to which i is currently linked and remove the edge (i, j). We refer to this new graph as $\mathbf{X_{t+1}}$. We refer to this step as a *transition* and the new graph is potentially nearer

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to achieving the set of inequalities on the degrees. Here we restrict ourselves (in the main) to the case where $m_i = M_i$ for all i, refer to the sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{S_n}$ where $a_i = m_i = M_i$ as the *target*, and to a_i as i's target.

1.1. Definitions

Definition 1. For two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{S_n}$ the **Distance** $z(\mathbf{u}, \mathbf{v}) = \Sigma_{i=1}^{n} |u_i - v_i|$, i.e. z is the distance generated by the L_1 norm.

Recall that $\mathbf{H_n} \subset \mathbf{S_n}$ is the set of graphic sequences. Now suppose that we consider target $\mathbf{a} \in \mathbf{S_n}$. We are interested in the distances between $\mathbf{a}$ and the graphic sequences, $\mathbf{H_n}$. We introduce the notion of a score of a sequence.

Definition 2. The **Score** of $\mathbf{a}$ is $s(\mathbf{a}) = min_{\mathbf{u} \in \mathbf{H_n}} z(\mathbf{a}, \mathbf{u})$.

Thus $s : \mathbf{S_n} \to \{0, 1, \ldots, k_n\}$, where $k_n = \lceil n/2 \rceil \lfloor n/2 \rfloor$ which occurs for the cases where we have $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$ targets $n-1$ and the remainder 0. We prove that the transitions for a particular target $\mathbf{a}$ lead to a subset of the possible sequences with minimal distance from the target. We refer to this as the minimal set of $\mathbf{a}$ defined as follows.

Definition 3. The **Minimal Set** of $\mathbf{a}$ is $\mathbf{J}(\mathbf{a}) = \{\mathbf{u} \in \mathbf{H_n} \mid z(\mathbf{u}, \mathbf{a}) = s(\mathbf{a})\}$.

Although we are focusing on the situation where we have $m_i = M_i$ for every vertex, we relax this condition for the moment as Theorems 1 and 2 are valid in the more general case. For this purpose we introduce the term deviation to take the place of our distance/score in the restricted case.

Definition 4. If the degree of vertex i is e_i, then we define the **Deviation** ϵ_i of that vertex by $\epsilon_i = max[(m_i - e_i), (e_i - M_i), 0]$.

Definition 5. The **deviation of the graph** $\mathbf{X_t}$ is defined as $\Upsilon_t = \Sigma_{i=1}^{n} \epsilon_i$.

We refer to a vertex as a Neutral, a Joiner or a Breaker, according as ϵ_i is 0, $m_i - e_i$ or $e_i - M_i$.

For a target $\mathbf{a}$, it is clear that the deviation of a graph $\mathbf{x}$ is the same as the distance of its associated sequence $\mathbf{u}$ from the target, and so the minimum deviation from the target over all graphs is simply the score.

1.2. Paper structure

We begin in Section 2 by introducing a population model in which the vertices are individuals and the targets correspond to the numbers of links which individuals would wish to maintain. This model has been discussed in detail in [1]. We then introduce the *Transition Graph*, which links graphs under the transitions defined for a given target. Using this concept we prove that the system, for a given target, possesses a set of graphs which have minimal deviation from the target (in a well defined sense).

This section is central to understanding the underlying population aspects that constitute the applications of our model, but is not necessary for any readers only interested in the graph-theoretical aspects, who can skip it and move straight to the sections that follow, starting with Section 3.

Section 3 considers the structure of minimal graphs with respect to a specific target. Any such minimal graph has each vertex classified as a Joiner, a Breaker or Neutral. Our fundamental result is Theorem 4, which demonstrates that for a given sequence the minimal graphs are characterized by the absence of certain patterns of presence and absence of edges. From this we can deduce certain local patterns and demonstrate a partition of the vertices in minimal graphs, as shown in Theorem 7. These ideas may help identify and enumerate the possible sets of minimal graphs.

Section 4 demonstrates a method for finding the minimal score for a given sequence. We address the question of finding the minimal score for a given target, and do this through a variant of the classical Havel–Hakimi algorithm [2,4], see Theorem 8. We finally address the issue of identifying all the minimal graphs for a given target using the Ferrer diagram [7], and exploiting the Ruch–Gutman [6] conditions, in Section 5. We prove that the set of Ferrer diagrams corresponding to minimal graphs for a given target is connected under valid transitions (Theorem 13), and moreover that the set of minimal graphs is connected (Theorem 14) except possibly when the score is zero.

2. Sequences, transitions and a Markov process

2.1. Markov Chain model

We introduce here a population model; there are n individuals, represented by the vertices of a graph, where individual i wishes to link to between m_i and M_i others. Supposing at time t that the graph is $\mathbf{X_t}$, and that individual i is selected with probability p_i. A transition is then made, as described earlier, where the edge to add or subtract is selected at random with equal probabilities. The process $\mathbf{X_t}$ is a homogeneous Markov chain with the following transition probabilities:

(1) For any graph $\mathbf{x}^*$, with sequence $\mathbf{u}$, which differs from $\mathbf{x}$ in a single entry, where $x_{ij} = 0$, $x^*_{ij} = 1$ for some i, j,

$$P(\mathbf{X_{t+1}} = \mathbf{x}^* \mid \mathbf{X_t} = \mathbf{x}) = \begin{cases} p_i \dfrac{1}{n-1-u_i} + p_j \dfrac{1}{n-1-u_j} & u_i < m_i, u_j < m_j \\ p_i \dfrac{1}{n-1-u_i} & u_i < m_i, u_j \geq m_j \\ p_j \dfrac{1}{n-1-u_j} & u_i \geq m_i, u_j < m_j \\ 0 & u_i \geq m_i, u_j \geq m_j. \end{cases}$$

(2) For any $\mathbf{x}^*$ which differs by $\mathbf{x}$ in a single entry, where $x_{ij} = 1$, $x^*_{ij} = 0$ for some i, j,

$$P(\mathbf{X_{t+1}} = \mathbf{x}^* \mid \mathbf{X_t} = \mathbf{x}) = \begin{cases} p_i \dfrac{1}{u_i} + p_j \dfrac{1}{u_j} & u_i > M_i, u_j > M_j \\ p_i \dfrac{1}{u_i} & u_i > M_i, u_j \leq M_j \\ p_j \dfrac{1}{u_j} & u_i \leq M_i, u_j > M_j \\ 0 & u_i \leq M_i, u_j \leq M_j. \end{cases}$$

(3) For any $\mathbf{x}^*$, differing from $\mathbf{x}$ in two or more entries, $P(\mathbf{X_{t+1}} = \mathbf{x}^* \mid \mathbf{X_t} = \mathbf{x}) = 0$.
(4) For $\mathbf{x}^* = \mathbf{x}$, the probability is 1 minus the sum of the above probabilities.

This model has been discussed by [1] and further work is in progress.

2.2. The deviation

Theorem 1. *For any set of ranges $[m_i, M_i]$ Υ_t is non-increasing, under the possible transitions, with t.*

Proof. Consider the transition from time t to time $t+1$. When a vertex i is selected then if it is Neutral no change happens so $\Upsilon(G_{t+1}) = \Upsilon(G_t)$. If it is a Joiner then its deviation is decreased by 1 (since an edge is added), and if that edge is joined to vertex j then the deviation of that vertex changes by -1, 0 or $+1$ according as (a) $u_j < m_j$ so j is a Joiner, (b) $m_j \leq u_j \leq M_j - 1$ so j is a Neutral or (c) $u_j = M_j$ so j is a Neutral, or $M_j < u_j$ so j is a Breaker. Thus $\Upsilon(G_{t+1}) - \Upsilon(G_t)$ is respectively -2, -1 or 0. A similar argument applies if we pick a Breaker. □

Corollary 1.1. *When $m_i = M_i$ all i then $\Upsilon(G_{t+1}) - \Upsilon(G_t)$ is -2 or 0.*

Proof. Case (b) from the proof of Theorem 1 is then impossible. □

Corollary 1.2. *If there is a path of transitions from a graph state $\mathbf{x}$ to another with a lower deviation, then $\mathbf{x}$ is a transient state, and so $P(\mathbf{X_t} = \mathbf{x}) \to 0$.*

The state space can be partitioned into a set of transient states, and some number of connected closed sets $S_1, S_2, \ldots, S_k$, the absorbing sets.

In Section 2.3 we define the transition graph. This is a directed graph and the sets $S_1, S_2, \ldots, S_k$ are the strongly connected subsets of that graph.

Corollary 1.3. *Each graph within a closed set has the same deviation.*

The states within each set communicate (i.e. any such state can be reached from any other) since the set is connected, so by Theorem 1 the states within any set have the same deviation, denoted by $\Upsilon(S_i)$. Since there are only a finite set of sets there is a minimum value. Our next theorem establishes that each set achieves this minimal deviation, which we denote by d_m.

Theorem 2. *The deviation of any graph in any of the absorbing sets is d_m.*

Proof. Suppose G_1 and G_2 have $\Upsilon(G_1) > \Upsilon(G_2)$.

We define $\delta_{ij} = 0$ if G_1 and G_2 both have the edge (i, j), or neither of them do, and else define $\delta_{ij} = 1$. We then define $\Delta = \Sigma_{i,j}\delta_{ij}$. We attempt to decrease Δ by making a valid transition from G_1.

Consider a Joiner, i say, in G_1. Suppose we can add an edge (i, j) to G_1 which is present in G_2, then Δ will decrease. We will be unable to do this only if all (i, j) not in G_1 are also absent from G_2, which implies that i is a Joiner in G_2 with a deviation at least as large as in G_1.

In a similar manner we consider a Breaker, i say, in G_1. If we can remove an edge (i, j) from G_1 which is absent from G_2 then Δ will decrease. We will be unable to do this only if all (i, j) in G_1 are also present from G_2, which implies that i is a Breaker in G_2 also with a deviation at least as large as in G_1.

Thus we can reduce Δ unless every i has a deviation at least as large in G_2 as in G_1, which contradicts the assumption that $\Upsilon(G_2) < \Upsilon(G_1)$. Repeating the process eventually the score will be reduced to that of G_2, and picking G_2 such that $\Upsilon(G_2) = d_m$ yields the result. $\square$

2.3. The transition graph

A graph of interest is that describing the possible transitions between graphs $\mathbf{X_t}$ induced by a specific target $\mathbf{a}$. In order to do this we need to expand the set $\mathbf{H_n}$, which has non-increasing elements, to encompass all permutations of these orderings; call this set $\mathbf{H_n^*}$. Now given a target $\mathbf{a}$, we define an equivalence relation $\sim$ on $\mathbf{H_n^*}$. We write $P(u)$ for the action of the permutation operator P on u. Suppose $\mathbf{Y}$ is the set of permutations which leave $\mathbf{a}$ unchanged, then for $u \in \mathbf{H_n^*}$ and $v \in \mathbf{H_n^*}$, $u \sim v$ if and only if for all $P \in \mathbf{Y}$, $u = P(v)$. We then choose a set of representatives $\mathbf{H_n^\dagger}$, that is one element from each equivalence class, which will correspond to the vertices of our transition graph.

Suppose that $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{H_n}$ are such that for specific i and j we have $u_i = v_i + 1$, $u_j = v_j + 1$ and $u_k = v_k$ otherwise. Now under the model for transitions described earlier a transition is possible from $\mathbf{u}$ to $\mathbf{v}$ if $u_i < a_i$ or $u_j < a_j$, and from $\mathbf{v}$ to $\mathbf{u}$ if $v_i > a_i$ or $v_j > a_j$. We define the transition graph $\mathbf{T_n} = \{\mathbf{H_n^\dagger}(a), \mathbf{F_n}(\mathbf{a})\}$ where $\mathbf{F_n}(\mathbf{a})$ is the set of possible transitions defined above; these edges are directed.

Given $\mathbf{a}$, associated with each vertex $\mathbf{u}$ in $\mathbf{T_n}(\mathbf{a})$ is the distance $d(\mathbf{u}, \mathbf{a})$. The transitions are of two types, those where the degrees of both the vertices which are affected move towards their target, so the score drops by 2, and ones where one moves towards and one away from its target when the score does not change. If, and only if, the transition is of the second type then there is also a transition in the reverse direction. Moreover, as we proved above in Theorem 2, it is always possible to move from a state to one with a lower score if such exists, though this may involve multiple steps. The system therefore will ultimately reach one of the graphic sequences u for which $d(\mathbf{u}, \mathbf{a})$ is minimal. If that score is zero, which only occurs when the target is itself graphic, then no further moves are possible. If the score is other than zero then the state can move around within $\mathbf{J}(\mathbf{a})$, and we prove that the graph induced by the vertices $\mathbf{J}(\mathbf{a})$ within $\mathbf{T_n}(\mathbf{a})$ is connected. This is not to assert that the set of corresponding graphs can all communicate; for example if $\mathbf{a} = (2, 2, 2, 2)$ so that $\mathbf{J}(\mathbf{a})$ has a single element $(2, 2, 2, 2)$ then no movement is possible, but there are three graphs corresponding to this degree sequence between which there is no communication.

The transition diagram can be obtained for given n and some specific target in the following way. We construct the set of possible permutationally distinct graphic sequences for that n and that partition. Then we join in the graph $\mathbf{T_n}(\mathbf{a})$ those sequences which differ by $+1$ in two positions, or by -1 in two positions. Note that this is always possible for any target, since clearly the target must differ from one of the sequences in the positions which are changed, and so there will be the possibility of incrementing or decrementing the position appropriately. Finally we can add the directions appropriate for the specific target by calculating the deviations, arrows going on every edge in the direction of non-increasing score.

3. The Breaker–Joiner structure of minimal graphs

This section addresses the question of the structure of minimal graphs. It is clear that every graph is the minimal graph for some non-empty set of targets since trivially it is a minimal graph whose target matches its degree sequence. It is also clear that it is a minimal graph for any target at distance 1, and also for any target at distance 2 provided that target is not itself graphic. A more general question is to characterize in some way the set of possible targets for which a given graph is a minimal graph.

We now introduce a fundamental entity which allows us to identify properties of minimal graphs. Essentially this is a sequence of vertices which allow us to make sequences of transitions in a graph, where all but the last keep the score unchanged and the final one improves the score. These transitions may not occur consecutively through time, but each time that a reduction in the score occurs we can trace back the set of transitions which led to that reduction. We encapsulate the possibilities in the following definitions.

Definition 6. An Alternating Edge Sequence (AES) for a graph $\mathbf{G}$ with edge set $\mathbf{E}$ is a sequence $\{v_1, v_2, \dots, v_r\}$ where $v_i \neq v_j$ for all $1 \leq i < j \leq r$ with the possible exception $v_1 = v_r$, and
(a) $(v_i, v_{i+1}) \in \mathbf{E}$ for i odd and $(v_i, v_{i+1}) \in \bar{\mathbf{E}}$ for i even or
(b) $(v_i, v_{i+1}) \in \bar{\mathbf{E}}$ for i odd and $(v_i, v_{i+1}) \in \mathbf{E}$ for i even.

Definition 7. An *Improvable Alternating Edge Sequence* (IAES) for a graph $\mathbf{G}$ with respect to a given target is an AES $\{v_1, v_2, \dots, v_r\}$, defined by the following conditions (here $\mathbf{J}$ is the set of Joiners in $\mathbf{G}$, $\mathbf{N}$ the set of Neutrals, and $\mathbf{B}$ the set of Breakers),
(1) if $1 < i < r$ then $v_i \in \mathbf{N}$,
(2) if $(v_1, v_2) \in \mathbf{E}$ then $v_1 \in \mathbf{B}$, else $v_1 \in \mathbf{J}$,
(3) if $(v_{r-1}, v_r) \in \mathbf{E}$ then $v_n \in \mathbf{B}$, else $v_n \in \mathbf{J}$.
In the case where $v_1 = v_r$ we require that r is even and that the deviation of $v_1 \geq 2$.

Theorem 3. *If $r > 2$ and $\{w_1, w_2, \dots, w_{i-1}, w_i, w_{i+1} \dots, w_{r-1}, w_r\}$ is an IAES then exactly one of $\{w_1, w_2, \dots, w_{i-1}, z\}$ and $\{z, w_{i+1}, \dots, w_{r-1}, w_r\}$ is an IAES where z is a Joiner with the same edge relationship to w_{i-1} and w_{i+1} as w_i. Similarly if z is a Breaker.*

Proof. The proof relies on the fact that if an IAES has ends of common type, i.e. both Joiners or both Breakers, then the number of terms is even, and if the ends are of distinct types then it is odd.

If w_1 and w_r are both Joiners, or both Breakers, then r is even, and hence one of $(r - i + 1)$ and i is odd and one even. Hence whether z is a Joiner or a Breaker, there will be an IAES in one direction from z, but not in the other. Note that if $w_1 = w_r$ so that the IAES is of type (3) in Definition 7, the result still holds.

If w_1 and w_r are different, i.e. one a Joiner and one a Breaker, then r is odd, and hence both of $(r - i + 1)$ and i are even, or both odd. Hence whether z is a Joiner or a Breaker, there will be an IAES in one direction from z, but not in the other. □

If **a** is graphic then there can exist no IAES's since every vertex is Neutral. For non-graphic sequences we have the following fundamental theorem.

Theorem 4. *A graph* **G** *is minimal with respect to some non-graphic target* **a** *if, and only if, there are no IAES's.*

Proof. (1) **G** is minimal implies that there are no IAES's.
Suppose **G** is minimal with respect to **a** and there exists an IAES. If $r = 2$ then we have an immediate improvement since v_1 and v_2 are both Breakers, or both Joiners. For $r > 2$ in each of the cases above we can make a valid transition using v_1 and v_2 with no change in the score, and following the transition the sequence $\{v_2, v_3, \ldots, v_r\}$ will be an IAES in the new graph. We make transitions until we are left with only v_{r-1} and v_r, and these vertices will be of the same type. Note that in the case where $v_1 = v_r$ the first vertex retains its type during the first transition since its score initially was ≥ 2. Thus the final transition will decrease the score by 2, which contradicts the fact that **G** is minimal. Thus there can have been no IAES's.

(2) There are no IAES's implies that **G** is minimal.
Suppose there is a target **a**, and a graph **G** which is non-minimal with respect to **a**. Suppose there are no IAES's.

Consider any transition, which clearly can make no change to the score since there are no Breaker–Breaker edges, or Joiner–Joiner non-edges. We prove that there will be no IAES subsequent to any change. Suppose without loss of generality that we delete an edge between vertex v_1, a Breaker, and vertex v_2 (adding an edge between a vertex v_1, a Joiner, and a vertex v_2 will have reciprocal consequences in the arguments below). Then v_2 must be a Joiner after the change.

Following the change there are various candidates to be IAES's; clearly only those which involve v_1, v_2 or both. We establish that the existence of an IAES after any switch implies that there was an IAES before the switch, a contradiction.

We know v_2 is a Joiner after the switch so must be at the end of any IAES of which it is a part. We first consider the three possibilities in which v_2 is part of a new IAES.

(i) Consider any sequence $\{v_2, w_1, \ldots, w_r\}$, where $w_r \neq v_2$ and v_1 does not occur. If this is an IAES then it would have originally have been one if v_2 had originally been a Joiner, while if v_2 had been Neutral then $\{v_1, v_2, w_1, \ldots, w_r\}$ was already an IAES.

(ii) Consider any sequence $\{v_2, w_1, \ldots, w_r, v_2\}$, where r is even, which does not contain v_1, so after the switch v_2 is a Joiner with score ≥ 2. If before the switch v_2 had a score ≥ 2 then $\{v_2, w_1, \ldots, w_r, v_2\}$ was already an IAES. In the case where the score of v_2 was previously 1 then $\{v_2, w_1, \ldots, w_r, v_2, v_1\}$ was already an IAES.

(iii) Consider a sequence including both v_1 and v_2. We need to consider the two possibilities for the state of v_1 after the switch.

(a) Suppose v_1 is Neutral after the switch, then the candidate to be an IAES is of the form $\{v_2, \ldots, v_1, \ldots, w_r\}$, and thus by Theorem 3 either $\{v_2, \ldots, v_1\}$ or $\{v_1, \ldots, w_r\}$ was already an IAES.

(b) Suppose v_1 is still a Breaker, then the IAES must be $\{v_2, \ldots, v_1\}$, and the number of terms must be odd. If v_2 was originally a Joiner then $\{v_2, \ldots, v_1\}$ was already an IAES, whereas if it were neutral then $\{v_1, v_2, \ldots, v_1\}$ was already an IAES since we know that v_1 had a score of at least 2 before the switch since it remained a Breaker.

(iv) Consider any IAES which does not include v_2, and hence must include v_1. If this is $\{v_1, \ldots, w_r\}$ then v_1 is still a Breaker and so the same sequence was an IAES before the change. The same argument applies for $\{v_1, \ldots, w_r, v_1\}$. If this is $\{w_1, \ldots, v_1, \ldots, w_r\}$ then this requires that v_1 changed to Neutral, so either $\{w_1, \ldots, v_1\}$ or $\{v_1, \ldots, w_r\}$ was an IAES originally, by Theorem 3.

Thus no change can create a graph with an IAES, so no graph can be reached for which the score can be reduced and thus by the argument of Section 2.2, the graph is minimal. □

The above necessary and sufficient condition provides us with a variety of local conditions, which are illuminating regarding the structure of a minimal graph. We list some of these. In each case an IAES is key but is not made explicit.

Corollary 4.1. *For any minimal graph the following properties hold.*
(1) *No two Breakers are joined.*
(2) *Every pair of Joiners are joined.*
(3) *If a Neutral is joined to a Breaker then that Neutral is joined to every Joiner* $\Leftrightarrow$ *if a Neutral is not joined to some Joiner then that Neutral is not joined to any Breaker.*
(4) *Suppose that a Neutral* N *is joined to some breaker* B*, and some other Neutral* N^* *is joined to some other Breaker* B^* *then* N *and* N^* *are joined.*
(5) *Suppose that some Neutral* N *is not joined to some Joiner, and another Neutral* N^* *is not joined to some other Joiner* J^* *then* N *is not joined to* N^**.*

Theorem 5. *For any target* **a** *the minimal set* **J**(**a**) *contains at least one member with no Joiners, and at least one with no Breakers.*

Proof. Consider some target **a**. If **a** is graphic then the set of minimal sequences consists of **a** alone, and this will have no Breakers and no Joiners so the theorem holds in this case.

Now suppose **a** is not graphic. We now prove that there exists a minimal graph which has only Breakers and/or Neutrals with respect to **a**. Choose any minimal graph with at least one Joiner. If this is not possible then, since the set of minimal graphs cannot be empty, there is a minimal graph with no Joiners. For the case where there exists such a minimal graph we successively add edges to the Joiners. This will be possible provided there is a Joiner which is not joined to every other vertex. This is always the case since a Joiner joined to every other vertex would imply a degree, for that Joiner, of $n-1$ which is the maximal possible degree and hence contradicts the fact that the vertex is a Joiner. We can thus proceed until all Joiners have been removed.

The argument to establish that there is a minimal graph with no Breakers proceeds in a similar manner removing edges from Breakers and observing that a vertex with degree 0 cannot be a Breaker. □

Theorem 6. *For a minimal graph the degree of a Joiner is greater than or equal to that of a Breaker.*

Proof. Consider the set consisting of all the Joiners and all of the Neutrals which are linked to a Breaker; these latter are linked to every Joiner. Denote this set by **H** and $|\mathbf{H}| = k$. Now we know from (2) and (3) of Corollary 4.1 that each Joiner is linked to every other element of **H**. Thus the degree of every Joiner is at least $k-1$; it may be larger since it may be linked to Breakers and to other Neutrals which are not themselves linked to any Breaker. Now each Breaker is linked only to some subset of **H**, by Corollary 4.1, so its degree is less than or equal to k. If it is equal to k then it is linked to every Joiner which therefore themselves have degree at least k, the $k-1$ links within **H** and the link to the Breaker, otherwise the Breaker's degree is at most $k-1$. □

Corollary 6.1. *With respect to the ordered target the Joiners precede the Breakers for any minimal graph.*

Proof. Since the degree of any Joiner is at least as large as that of any Breaker, and the target of a Joiner is greater than the degree, and that of a Breaker is less, the target of a Joiner must strictly exceed that of any Breaker. □

It is intuitively reasonable that those vertices (individuals) with higher targets will fall short while those with lower targets will exceed them.

From the above we have the following structure for a Minimal Graph.

Theorem 7. *For a minimal graph with respect to a specific target the vertices can be partitioned into five subsets (some of which may be empty):-*

V1 — a complete subgraph which contains the set of Joiners,
V2—an independent subset which contains the set of Breakers,
V3 — a subset of Neutrals each of which has at least one link to a Breaker and each is linked to all the Joiners,
V4 — a subset of Neutrals which are not linked to every Joiner and each is not linked to any Breaker,
V5 — a subset of Neutrals each of which has no links to any Breakers and is linked to every Joiner.

Proof. This follows directly from Corollary 4.1.

Fig. 1 illustrates the partitioning for the target {8886632211}. □

4. Finding the minimal deviation

In order to find the minimal deviation d_m we could examine all possible graphs for given n and evaluate their deviations. However, there are more efficient methods which give the deviation for any specified target (i.e. where for all i, $m_i = M_i$).

We shall give two simple methods of deriving the minimal deviation. The first is based on the algorithm of Havel [4] and Hakimi [2] which allows one to check whether a given sequence is graphic, and also generates an example of such a graph. The second method exploits the ideas of Hässelbarth and Ruch–Gutman.

4.1. The Havel–Hakimi algorithm

Given a sequence one applies an algorithm introduced by Havel [4] and Hakimi [2], which either produces a simple graph with the desired degree sequence, or fails in which case the sequence is not graphic.

The Havel–Hakimi algorithm is as follows. Suppose one has a sequence of degrees $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$ where $d_j \geq d_{j+1} \geq 0$ for $1 \leq j \leq (n-1)$. If $d_1 > 0$ and $d_{d_1+1} \leq 0$ then the sequence is not graphic. Assuming $d_1 > 0$, if $d_{d_1+1} \geq 1$ then replace the degree sequence by $\{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n\}$; essentially connect the vertex with highest degree to those others with next highest degrees, then eliminate d_1 from the sequence, reorder the new sequence (if necessary) in decreasing order, and take this as the new sequence which we need to check to see if it is graphic. Apply this process recursively. If this can be done until one arrives at a sequence consisting entirely of 0's then the sequence is graphic.

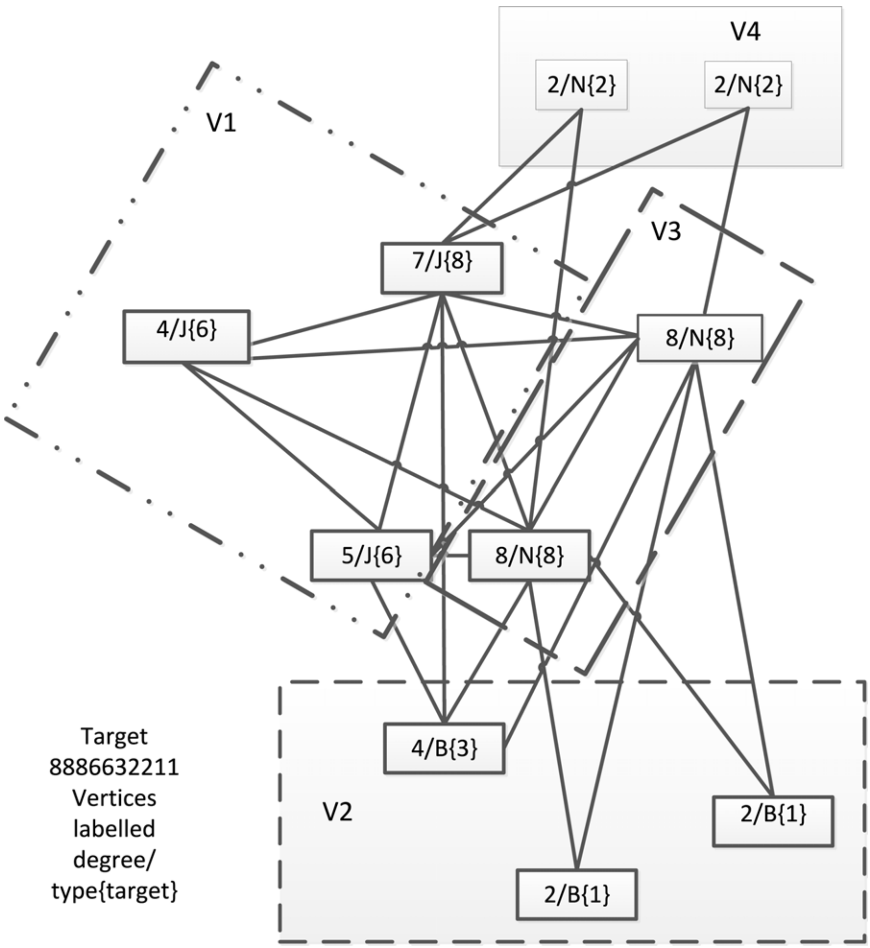

Fig. 1. A minimal graph for the target 8886632211 separating the Joiners V1, the Breakers V2, Neutrals joined to at least one Breaker and thus all Joiners V3, and Neutrals not joined to any Breaker V4. The numbering is as in Theorem 7; there is no set V5.

4.2. Reordered Havel–Hakimi algorithm

We introduce a new algorithm which is a slight modification of the Havel–Hakimi algorithm which has an easier application in subsequent proofs. In the Havel–Hakimi algorithm one reduces the elements in positions 2 through $d_1 + 1$ by one, and then reorders to obtain a decreasing sequence. Reordering is required if there is a substring of the original sequence, **s**, of more than one equal elements which begins at, or before, the $d_1 + 1$'st element and extends beyond it. Here we avoid the reordering step in the above case by instead subtracting ones from the elements at the end of this string **s**.

Formally, suppose one has a sequence of degrees $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$ where $d_j \geq d_{j+1}$ for $1 \leq j \leq (n-1)$. If $d_{d_1+1} \leq 0$ then the sequence is not graphic. Assuming $d_1 > 0$, if $d_{d_1+1} \geq 1$ specify e and f such that $d_{d_1-e+1} > d_{d_1-e+2} = d_{d_1-e+3} \cdots = d_{d_1+1} = d_{d_1+2} = \cdots = d_{d_1+f+1} > d_{d_1+f+2}$. Note that $e \geq 1$ is the number of terms with value d_{d_1+1} which would be decremented in the Havel–Hakimi algorithm and $f \geq 0$ is the number of such values beyond the d_{d_1+1} term. Now replace the degree sequence by $\{d_2 - 1, d_3 - 1, \ldots, d_{d_1-e+1} - 1, d_{d_1-e+2}, d_{d_1-e+3} \ldots d_{d_1-e+1+f}, d_{d_1-e+2+f} - 1, d_{d_1-e+3+f} - 1 \ldots d_{d_1+1+f} - 1, d_{d_1+2+f}, \ldots, d_n\}$.

It is clear that the above algorithm provides necessary and sufficient conditions for a graphic sequence in the same way as the Havel–Hakimi algorithm (although it does not necessarily produce the same graph).

4.3. Variant of Havel–Hakimi and reordered Havel–Hakimi algorithm

We now introduce a variant which delivers the minimal deviation of the sequence. If we have non-increasing sequence $\{d_1, d_2, \ldots, d_n\}$ at some stage, then suppose $k = min_i\{d_i = 0\}$, if $k < d_1 + 1$, then add 1 to the elements d_i for $k \leq i \leq d_1 + 1$ (note that this necessarily preserves the ordering), and proceed with the H–H or reordered H–H algorithm. The algorithm will not fail at any stage, and will generate a sequence of sets of vertices to which 1's have been added.

Definition 8. The **Havel–Hakimi Score** is the total number of 1's added during the variant algorithm.

For a sequence **u** we will denote the **Havel–Hakimi Score** by $s_{HH}(\mathbf{u})$.

We shall prove below that the H–H score is the score of the sequence.

We should note also that the sets of 1 which are added during the algorithm allow us to define a sequence of modified sequences, the final one of which is graphic. Suppose that if at some stage of the algorithm we need to add 1's to a number of vertices for $i \in T$ then we define a vector $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ where $t_i = 1$ for $i \in T$ and 0 otherwise. If $T = \phi$ then we have a vector of 0's. Note that the 1's are added to consecutive elements of the sequences during the algorithm, and that it will never subsequently affect vertices to which 1's need to be added, as these are always 0 at the end of the sequence, and will still be 0's after the removal of the leading term. Suppose that $\mathbf{w_r}$ is the sum of the vectors **t** used in the first r steps of the algorithm, and m_r is the sum of the elements of $\mathbf{w_r}$. Then $\mathbf{d} + \mathbf{w_r}$ has deviation $d_m - m_r$.

Theorem 8. *The HH-score equals the score of the sequence; i.e. the modified algorithm produces a minimal graph.*

Proof. Consider the set S_n of nonnegative integer sequences of length n. For $\mathbf{u}, \mathbf{v} \in S_n$ recall $z(\mathbf{u}, \mathbf{v}) = \Sigma_i |u_i - v_i|$.

Consider one of the algorithms above. For $u \in S_n$ this will produce a unique value, defining the function $s_{HH}(u)$. Now if for some v $z(\mathbf{u}, \mathbf{v}) = 1$ then the vectors **u** and **v** are identical in every entry except one, where their elements differ by 1. Suppose w.l.o.g. that $v_k = u_k + 1$. Applying the algorithm either (i) $k = 1$ and subsequently the vectors are identical except at index $1 + v_k$ where they differ by 1, (ii) $k > 1$, $v_k < v_{k-1}$ and subsequently the vectors are identical except at index k where they differ by 1, (iii) $k > 1$, $v_k = v_{k-1}$ and subsequently the vectors are identical except at precisely one index between 2 and k. Thus the difference of 1 is preserved. This process is repeated until a zero term has a 1 added in one sequence and not the other (sometimes there will be an identical number of zeros with one added in each sequence) after which the sequences will be identical, but with one more 1 added to one than the other. Thus $|s_{HH}(u) - s_{HH}(v)| = 1$.

It follows that for all $\mathbf{u}, \mathbf{v} \in S_n$, $z(\mathbf{u}, \mathbf{v}) \geq |s_{HH}(\mathbf{u}) - s_{HH}(\mathbf{v})|$

since if we have $\mathbf{u}, \mathbf{v} \in S_n$ with $z(\mathbf{u}, \mathbf{v}) = k$

there is a path $\mathbf{u} = \mathbf{z_1}, \mathbf{z_2}, \ldots, \mathbf{z_{k-1}}, \mathbf{v} = \mathbf{z_k}$ where $z(\mathbf{z_i}, \mathbf{z_{i+1}}) = 1$, and

$$z(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{k-1} z(\mathbf{z_i}, \mathbf{z_{i+1}}) = \sum_{i=1}^{k-1} |s_{HH}(\mathbf{z_i}) - s_{HH}(\mathbf{z_{i+1}})| \geq |\sum_{i=1}^{k-1} (s_{HH}(\mathbf{z_i}) - s_{HH}(\mathbf{z_{i+1}}))|$$

with equality iff all $(s_{HH}(\mathbf{z_i}) - s_{HH}(\mathbf{z_{i+1}})) = 1$ or all $(s_{HH}(\mathbf{z_i}) - s_{HH}(\mathbf{z_{i+1}})) = -1$. Now any graphic sequence **w** has $s_{HH}(\mathbf{w}) = 0$, so $z(\mathbf{w}, \mathbf{u}) \geq s_{HH}(\mathbf{u})$ and since $d(HH(\mathbf{u}), \mathbf{u}) = s_{HH}(\mathbf{u})$ we have $HH(\mathbf{u})$ as a minimal graphic sequence for **u**. □

5. Hässelbarth and Ruch–Gutman

In this section we demonstrate how the graphs of the minimal set are related to certain Ferrer diagrams. Given a sequence $\mathbf{u} \in \mathbf{S_n}$ then its conjugate **v** is defined by $v_i = \#\{j : u_j \geq i\}$ (where "#" means "the number of") for $i = 1, \ldots, u_1$. Note that this is a bijection. Thus, following the example in [5], if $\mathbf{u} = (4, 3, 3, 2, 2, 2)$ then $\mathbf{v} = (6, 6, 3, 1)$. These quantities relate to the Ferrer diagram (see for example [7]), the lengths of the rows are equal to the u_i's, the conjugate **v** lists the lengths of the columns, while $\mathbf{f}(\mathbf{u}) = \#\{i : x_i \geq i\}$ is the length of the diagonal, which is referred to as the *Durfee Number*, since the largest square within the Ferrer diagram is called a Durfee Square.

Suppose we have some **u**, and hence $\mathbf{f}(\mathbf{u}) = \lambda$ say, and **v**. Now define $\mathbf{u}^{\boldsymbol{\lambda}} = \{u_1, u_2, \ldots, u_\lambda\}$ and $\mathbf{v}^{\boldsymbol{\lambda}} = \{v_1, v_2, \ldots, v_\lambda\}$. This pair $\{\mathbf{u}^{\boldsymbol{\lambda}}, \mathbf{v}^{\boldsymbol{\lambda}}\} \Leftrightarrow \mathbf{u}$, and are sometimes easier to work with.

Theorem 9 *([6]). Ruch–Gutman Theorem:-if* **u**, *with conjugate vector* **v**, *is such that* $\Sigma_i u_i$ *is even, then* **u** *is graphic if, and only if,* $\Sigma_{i=1}^k u_i \leq \Sigma_{i=1}^k (v_i - 1)$, $1 \leq k \leq \mathbf{f}(\mathbf{u})$.

Definition 9. If **a** and **b** are n-vectors with elements arranged in decreasing size, then **a** is said to **majorise b** [7], which we write as $\mathbf{a} \succ \mathbf{b}$, if for all $1 \leq m \leq n$ we have $\Sigma_{i=1}^m a_i \geq \Sigma_{i=1}^m b_i$.

In terms of majorisation we can state the Ruch–Gutman theorem, as the following:-

Theorem 10. *Ruch–Gutman Theorem:- A sequence* **u** *is graphic if* $\Sigma_i u_i$ *is even and* $\mathbf{v}^{\boldsymbol{\lambda}} \succ \mathbf{u}^{\boldsymbol{\lambda}} + \mathbf{1}^{\boldsymbol{\lambda}}$, *where* $\mathbf{1}^{\boldsymbol{\lambda}}$ *is the unit vector of length* λ.

Definition 10. The **Deficit Vector for a target u** is a vector $\mathbf{d} = \{d_1, d_2, \ldots, d_\lambda\}$. Suppose $\mathbf{e} = \{e_1, e_2, \ldots, e_\lambda\}$ where $e_i = max_{j \leq i}[0, \Sigma_{r \leq j}(u_r + 1 - v_r)]$, so the e_i are the nonnegative record values. Now define $d_i = e_i - e_{i-1}$, for $i = 1, \ldots, \lambda$ where we take $e_0 = 0$.

Definition 11. The **Extreme Vectors for a Target u** are $\mathbf{v} + \mathbf{d}$ and its conjugate. We write these as $\mathbf{v}^*$ and $\mathbf{u}^*$ and refer to them as the extreme v-vector and the extreme u-vector.

Definition 12. The **Deficit** is the sum $\Sigma_i d_i$.

The deficit for a target is necessarily equal to the score defined earlier, and to the HH-score.

Definition 13. For a specific target $\mathbf{u}$, there being n elements, a vector $\mathbf{d} = \{d_1, d_2, \ldots, d_\lambda\}$ is said to be **acceptable for u** if $v_1 + d_1 \leq (n-1)$ and the elements of $\mathbf{v^\lambda} + \mathbf{d^\lambda}$ are non-increasing.

Definition 14. The **Deficit Set for some target u** is the set of vectors $\{\mathbf{T(u)} = \{\mathbf{z} = \{z_1, z_2, \ldots, z_\lambda\}, \Sigma_i^\lambda z_i = \Sigma_i^\lambda d_i | \mathbf{z} \succ \mathbf{d}\ ;\ \mathbf{z}$ **acceptable for u**$\}$, where $\mathbf{d}$ is the deficit vector for $\mathbf{u}$.

The deficit set $\mathbf{T(u)}$ identifies the set of Ferrer diagrams for which the corresponding graph has a deviation equal to the deficit for the specified target, and in which only Breakers are present. Corresponding to each element of $\mathbf{T(u)}$ there is an n-element vector $\mathbf{v}$ and its conjugate $\mathbf{u}$. Note that given $\mathbf{u}$ and its conjugate $\mathbf{v}$, if the conjugates of $\mathbf{u}+\mathbf{g}$ and of $\mathbf{u}+\mathbf{h}$, are $\mathbf{v}+\mathbf{r}$ and $\mathbf{v}+\mathbf{s}$, then $r \succ s \Leftrightarrow h \succ g$.

5.1. The structure of *T(u)*

Theorem 11. *If for some* $\mathbf{u}$ *we have corresponding deficit set* $\mathbf{T(u)}$*, then if* $\mathbf{z} \in \mathbf{T(u)}$ *and* $\mathbf{z^*} \in \mathbf{T(u)}$*, where* $\mathbf{z} \succ \mathbf{z}^*$*, there exists a sequence* $\{\boldsymbol{\zeta}_\mathbf{i}\}$ *with* $\boldsymbol{\zeta}_\mathbf{i} \in \mathbf{T(u)}$ *for* $i = 1, 2, \ldots, r$*, where* $\boldsymbol{\zeta}_\mathbf{0} = \mathbf{z}$*,* $\boldsymbol{\zeta}_\mathbf{r} = \mathbf{z}^*$*, and* $\boldsymbol{\zeta}_\mathbf{i+1} - \boldsymbol{\zeta}_\mathbf{i} \in \boldsymbol{\Delta}$ *the set of vectors of length n where there are* $n-2$ *elements equal to* 0*, one equal to* -1 *which occurs earlier than one equal to* $+1$.

Proof. Suppose we have distinct $\boldsymbol{\phi} \in \mathbf{T(u)}$ and $\boldsymbol{\phi}^* \in \mathbf{T(u)}$, where $\boldsymbol{\phi} \succ \boldsymbol{\phi}^*$. We observe first that since $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^*$ are distinct and $\boldsymbol{\phi} \succ \boldsymbol{\phi}^*$ it follows that $\phi_i = c$ for all i and some c is not possible, and that if $\phi_i = a, i \leq j$ and $\phi_i = b, j < i \leq n$, then $a \geq b+2$.

Define $y = min_i\{(\phi_i > \phi_i^*) \bigcap (\phi_i > \phi_{i+1})\}, z = min_i\{(i > y) \bigcap (\Sigma_{j=1}^i \phi_j = \Sigma_{j=1}^i \phi_j^*)\}$, which implies that $\phi_z < \phi_z^*$, and $x = min_i\{(z \geq i > y) \bigcap (\phi_i = \phi_z)\}$. We have that $y < x, \phi_{y-1} \geq \phi_y > \phi_{y+1}, \phi_y > \phi_y^*, (\Sigma_{i=1}^j \phi_i > \Sigma_{i=1}^j \phi_i^*)\ \forall y < j < x$, $\phi_{x-1} > \phi_x \geq \phi_{x+1}$ and $\phi_x < \phi_x^*$. Note that in the case where $y = x-1$ we have that $\phi_y \geq \phi_x + 2$. These conditions imply that $f(\boldsymbol{\phi}, \boldsymbol{\phi}^*) = \boldsymbol{\phi} - \boldsymbol{\delta}_\mathbf{y} + \boldsymbol{\delta}_\mathbf{x}$ is acceptable and that $\boldsymbol{\phi} \succ f(\boldsymbol{\phi}, \boldsymbol{\phi}^*) \succ \boldsymbol{\phi}^*$.

Suppose now we take $\boldsymbol{\zeta}_\mathbf{0} = \mathbf{z}$ and $\zeta_i = f(\boldsymbol{\zeta}_\mathbf{i-1}, z^*)$ for $i > 1$ then for some r we will obtain $\boldsymbol{\zeta}_\mathbf{r} = \mathbf{z}^*$, as required. □

Further it is clear that there is no $\mathbf{s} \in \mathbf{T(u)}$ with $\boldsymbol{\zeta}_\mathbf{j-1} \succ \mathbf{s} \succ \boldsymbol{\zeta}_\mathbf{j}$. Note also that the sequence defined in the proof is uniquely defined, though there may be other sequences $\{\boldsymbol{\chi}_\mathbf{i}\}$ with $(\boldsymbol{\chi}_\mathbf{i} - \boldsymbol{\chi}_\mathbf{i+1}) \in \boldsymbol{\Delta}$ which run from $\mathbf{z}$ to $\mathbf{z}^*$, necessarily with r terms. The value of r is simply half the distance between $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^*$ since the distance is reduced by 2 at each step.

Corollary 11.1. *In the notation of Theorem 11 suppose* $\mathbf{z}^* = \mathbf{u}^*$*, where* $\mathbf{u}^*$ *is the extreme u-vector, then the above process defines for each* $\mathbf{z} \in \mathbf{T(u)}$ *a unique sequence in which each element covers the next, starting from* $\mathbf{z}$*. There is a corresponding sequence* $\mathbf{v} + \boldsymbol{\zeta}_\mathbf{i}$ *and the conjugates of these* $\boldsymbol{\eta}_\mathbf{i}$ *say, have* $\boldsymbol{\eta}_\mathbf{i+1} \succ \boldsymbol{\eta}_\mathbf{i}$ *and* $\boldsymbol{\eta}_\mathbf{r} = \mathbf{u}^*$*, and* $\boldsymbol{\eta}_\mathbf{i} - \boldsymbol{\eta}_\mathbf{i+1} \in \boldsymbol{\Delta}$.

Example 1. Suppose we have $\mathbf{z} = \{77666433333311\}$ and $\mathbf{z^*} = \{77654444432222\}$; note that $\Sigma_{i=1}^j z_i = \Sigma_{i=1}^j z_i^*$ for $j = 9$ and $j = 14$. This implies that the earlier iterations deal with the first 9 elements and then later iterations deal with the final 5. The distance between $\mathbf{z}$ and $\mathbf{z^*}$ is 10 so 5 steps are required. The steps define a sequence $\{77666433333311\} - > \{77665443333311\} - > \{77655444333311\} - > \{77654444433311\} - > \{77654444433221\} - > \{77654444432222\}$.

5.2. Notation

Rather than write out $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ we shall specify $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ where the x_i's are the lengths of the downward runs in the Ferrer diagram progressing from the right to the left, and $\mathbf{y} = \{y_1, y_2, \ldots, y_m\}$ where the y_i's are the lengths of the sideways runs in the Ferrer diagram progressing from the lower to the upper part. Thus for the simple example above where $\mathbf{u} = \{4, 3, 3, 2, 2, 2\}$ we have $\mathbf{x} = \{1, 2, 3\}$ and $\mathbf{y} = \{2, 1, 1\}$. Now if we write $\boldsymbol{\nu} = \{\nu_1, \nu_2, \ldots, \nu_m\}$ where $\nu_i = \Sigma_{j=1}^{m+1-i} y_j$ then we write $\boldsymbol{\nu}^\mathbf{x} = \{\nu_1^{x_1}, \nu_2^{x_2}, \ldots, \nu_m^{x_m}\}$. Thus for the particular $\mathbf{u}$ here we can write $\boldsymbol{\nu}^\mathbf{x} = \{4^1, 3^2, 2^3\}$, which by a slight abuse of notation we shall denote as $\mathbf{u}$. Similarly if $\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots, \tau_m\}$ where $\tau_i = \Sigma_{j=1}^{m+1-i} x_j$ we can write the conjugate of $\mathbf{u}$ as $\boldsymbol{\tau}^\mathbf{y} = \{\tau_1^{y_1}, \tau_2^{y_2}, \ldots, \tau_m^{y_m}\}$, and so here we write $\mathbf{v} = \{6^2, 3^1, 1^1\}$.

Example 2. $n = 26$, $\mathbf{u} = \{25^2, 24^1, 20^5, 14^2, 9^8, 7^1, 6^3, 5^1, 4^1, 1^2\}$ (Note that the sum of the elements of $\mathbf{u}$ is even) and hence $\mathbf{v} = \{26^1, 24^3, 23^1, 22^1, 19^1, 18^2, 10^5, 8^6, 3^4, 2^1\}$. Note that we could have instead used $\lambda = 10$, $\mathbf{u^\lambda} = \{25^2, 24^1, 20^5, 14^2\}$ and $\mathbf{v^\lambda} = \{26^1, 24^3, 23^1, 22^1, 19^1, 18^2, 10^1\}$. Now we have $\mathbf{e} = \{0, 2, 3, 3, 3, 3, 3, 3, 3, 4\}$ and so the deficit vector $\mathbf{d} = \{0, 2, 1, 0, 0, 0, 0, 0, 0, 1\}$.

We now demonstrate how to generate all possible graphs with minimum score, and prove that this set is connected under the permitted transitions in Theorem 13. Our technique is to increase appropriately the elements of $\mathbf{v^\lambda}$, so that the Ruch–Gutman criterion is satisfied, and then form the conjugate of this modified vector. This will be a graphic sequence at distance d from the target, with a set of Breakers, and no Joiners. We can then proceed iteratively, see the algorithm below, to obtain all possible graphic sequences at minimal distance from the target.

Example 2 continued. $\mathbf{v}^\dagger = \mathbf{v}^\lambda + \mathbf{d} = \{26, 26, 25, 24, 23, 22, 19, 18, 18, 11\}$ which implies that the modified $\mathbf{u}$ becomes $\mathbf{u}^\dagger = \{25^2, 24^1, 20^5, 14^2, 10^*, 9^7, 7, 6^3, 5, 4, 3^*, 2^*\}$, where the elements which differ from $\mathbf{u}$ have been marked with "$*$". There are thus three Breakers (one with an excess of two) for this particular graphic sequence vis-a-vis the target. The process of selecting those elements of $\mathbf{T}(\mathbf{u})$ which are valid increments is not straightforward since it must be done while maintaining the restrictions on the vectors; not exceeding n and maintaining the ordering of the elements. Thus for example here we may not choose $\mathbf{d} = \{4, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$ since then $v_1 + d_1 = 29$ which exceeds n nor $\mathbf{d} = \{0, 3, 1, 0, 0, 0, 0, 0, 0, 0\}$ since we have $v_2 + d_2 = 27 > v_1 + d_1 = 26$ and the second element of $\mathbf{v}^\dagger$ exceeds the first, i.e. violates the required ordering.

5.3. Odds and evens

Recalling that the sum of the degrees of a graph is necessarily even we need to allow for this in our process. If the deficit is zero and the sum of the u_i is even then the target is a graphic sequence. If the deficit is odd and the sum is even, or if the deficit is even and the sum odd, then we need to adjust the deficit by one. The current example has even sum for $\mathbf{u}$ and even deficit so no adjustment is necessary.

Example 3. "Odd" total: $n = 10$, $\mathbf{u} = \{9, 8, 8, 6, 6, 5, 5, 3, 3, 2\}$ the total being odd.

Using the formulae given in Section 5.2 we have $\mathbf{u} = \{9^1, 8^2, 6^2, 5^2, 3^2, 2^1\}$ so $\mathbf{x} = \{1, 2, 2, 2, 2, 1\}$ and $\mathbf{y} = \{2, 1, 2, 1, 2, 1\}$ so $\mathbf{v} = \{10^2, 9^1, 7^2, 5^1, 3^2, 1^1\}$. Now the deficit vector is $\{0, 0, 0, 0, 0\}$ and the deficit 0. We need to adjust both $\mathbf{u}$ and $\mathbf{v}$ by one. We can change $\mathbf{u}_\lambda$ by a suitable reduction to an element of $\{9, 8, 8, 6, 6\}$, so the possibilities are $\{8, 8, 8, 6, 6\}$, $\{9, 8, 7, 6, 6\}$ or $\{9, 8, 8, 6, 5\}$, by a suitable increase to $\{9, 9, 8, 6, 6\}$ or $\{9, 8, 8, 7, 6\}$, or by a reduction to $\mathbf{v}$ to $\{10, 10, 9, 7, 6\}$, $\{10, 10, 8, 7, 7\}$ or $\{10, 9, 9, 7, 7\}$. Thus there are seven minimal graphs in this case.

5.4. An algorithm based on the Ferrer diagram

The set of graphs with minimal distance from the target can be generated sequentially from the deficit vector.

Having calculated the deficit vector we can generate the set of Ferrer diagrams corresponding to minimal graphs with no Joiners, as illustrated in the example above. We build up the complete set of Ferrer diagrams for all minimal graphs for the given target. Each operation corresponds to breaking one of the edges from a current Breaker, that is, using a standard transition step.

In these diagrams we differentiate between the points which belong to the target, referred to as target points, and those which have been added in accordance with the deficit, referred to as deficit points. Any point which has no other point below it and none to the right of it will be referred to as a corner; again we differentiate between target corners and deficit corners. At every stage we number the target corners with the number of the row in which it occurs, and the deficit corners with the number of the column in which it occurs, counting from the bottom left of the diagram. We move to another valid graph with minimal score by (1) removing a deficit corner with number $i \le \lambda$ and then (2) removing a target corner with number j with $i \le j \le \lambda$.

We now prove, in Theorem 13, that the set of Ferrer diagrams generated by the above algorithm contains all possible such diagrams for the specific target, that is, identifies all minimal graphic sequences.

Theorem 12. *Given any target* $\mathbf{u}$ *the set of sequences corresponding to the elements of* $\mathbf{T}(\mathbf{u})$ *are connected under valid transitions.*

Proof. We established in Theorem 11 that for any target $\mathbf{u}$ and $\mathbf{z} \in \mathbf{T}(\mathbf{u})$ if $\mathbf{u}^\dagger$ is the conjugate of $\mathbf{v}^* + \mathbf{z}$ there is a sequence of elements $\{\boldsymbol{\eta}_0 = \mathbf{u}^\dagger, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_\mathbf{r} = \mathbf{u}^*\}$ where $\boldsymbol{\eta}_\mathbf{i} - \boldsymbol{\eta}_{\boldsymbol{i}+\mathbf{1}} \in \boldsymbol{\Delta}$. We prove here that there is a realization of these sequences by constructing a sequence of graphs.

In the case where every Breaker is linked only to saturated vertices there is only one possible minimal graph with only Breakers, so there is nothing to prove.

Consider a graph $\mathbf{x}$ whose degree sequence is $\boldsymbol{\eta}_\mathbf{i}$. Suppose that in $\mathbf{x}$ there is a vertex k which is a Breaker and such that there is at least one vertex l, where $(k, l) \in \mathbf{E}$, which is not saturated; recall each vertex is either a Breaker or Neutral for elements of $\mathbf{T}(\mathbf{u})$. A valid transition will remove the edge (k, l) where l is Neutral in $\mathbf{x}$. In the resulting graph l is a Joiner and so a further valid transition will join l to some $j \neq k$ which then becomes, or remains, a Breaker. This new graph, $\mathbf{y}$, has degree sequence $\boldsymbol{\eta}_{\boldsymbol{i}+\mathbf{1}}$, differing from $\boldsymbol{\zeta}_\mathbf{i}$ by $+1$ in position j and by -1 in position k. Further the pair of transitions can be applied in reverse order from $\mathbf{y}$ to $\mathbf{x}$. It follows that any pair $\boldsymbol{\eta}_\mathbf{i}$ and $\boldsymbol{\eta}_{\boldsymbol{i}+\mathbf{1}}$ are connected in both directions and hence connected to $\mathbf{u}^*$. Thus all elements of $\mathbf{T}(\mathbf{u})$ are connected. □

Theorem 13. *The set of Ferrer diagrams for minimal graphs with respect to a target is connected under valid transitions.*

Proof. Given any Ferrer diagram of a minimal graph we have a corresponding specification of the Breakers and Joiners. We can repeatedly join vertices, using the Joiners, until we reach a graph with all Breakers. By Theorem 12 we have that these are connected, hence all Ferrer diagrams are connected. □

Theorem 14. *The set of minimal graphs is connected, except when the deviation is zero.*

Proof. We have proved that the set of Ferrer diagrams is connected under appropriate moves. However, the Ferrer diagram may correspond to more than one graph. We prove that if we have two distinct graphs with the same degree sequence then they are connected under the allowable transitions.

Suppose that we have two graphs which have no Joiners and with identical degree sequences. We will refer to these as the red and the blue graphs. From these we construct a new graph with the same vertex set and consisting precisely of edges which are present in one of the graphs but not the other, and with edges coloured as per the graph they belonged to. Now in this new graph each vertex has equal numbers of red and blue edges incident on it. First we consider the set of vertices which were Neutral in the original graphs. We take any such vertex and construct a path of alternating red and blue edges continuing until this path is closed by returning to the initial vertex, necessarily along a blue edge. We can repeat this operation until we have exhausted all the vertices and edges at which stage we have a set of alternatively coloured cycles.

We begin by demonstrating that we can make transitions to the red graph which change all the Neutral vertices to match those in the blue graph. We take a cycle. There are two cases to consider. (1) If this cycle has a vertex i joined to some Breaker in the red graph we break that link thus making i a Joiner and then proceed around the cycle first joining along the blue edge creating a Breaker while i becomes Neutral again, then along and removing the next red edge creating a Joiner and leaving a Neutral behind. This continues until i is reached again at which stage it is a Joiner and is then joined back to the original Breaker. We have removed all the blue edges of the cycle and created the red edges. (2) If the cycle has no vertex joined to the Breakers we then proceed as follows. Pick any Breaker and choose one of the Neutrals to which it is linked, break that link creating a Joiner which is then joined to the cycle in question, now proceed around the cycle and back to the Breaker through its link.

Repeating the above processes until there are no cycles remaining gives us a new red graph which has exactly the same subgraph on the Neutrals as the blue graph. The only edges differing now are from Breakers to Neutrals, and again we consider these as red and blue edges. Any Neutral has an equal number of red and blue edges to the set of Breakers. Choose a Neutral together with an incident red and blue edge. The Breaker on the red edge can break that edge and then the Joiner created can join along the blue edge. Repeating this step moves the red graph to the blue graph. □

6. Discussion

In this paper we have initiated the study of, what we have termed, graphic deviations; the distance of the nearest graphic sequences to some specific target sequence (or sequence set). We have addressed a number of problems but there are clearly many interesting issues which might be examined. Can we find the score for certain classes of targets? Can we say something about the graph of minimal graphs for a specific target, its size, diameter? Can we say something about the longest path from any graph to a minimal graph?

In the main we have discussed the situation in which each individual (vertex) has an exact target (desired degree). More generally each individual has a range of acceptable degrees, (m_i, M_i) for individual i. While Theorems 1, 2 and 5 apply in this general context, our later theorems do not. We shall present appropriate generalizations in a later paper.

The original motivation for considering this problem was the idea that within a population individuals may have different ideal numbers of links, and that individuals are repeatedly attempting to make or break links to get nearer to their target. We have proved here that the graph of links will under this scheme steadily approach a member of the set of minimal graphs, and then remain in that set. In [1] we discuss the Markov Chain which results when we attach a probability to the selection of the next individual who attempts to change their score. We discuss the limiting distribution for this process, demonstrating that the process is reversible which allows relative probabilities to be easily computed for the final states. As opposed to in this paper, where we are interested in whether given paths between states exist, in our Markov Chain model the probabilities involved (e.g. the probability that an individual is selected, the probability that it forms/ breaks a link) are important to the outcome of the process. As well as finding general results, we explore special cases such as when each individual is selected with equal probability, and asymptotic results where the population is large but the vast majority are Joiners (Breakers).

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