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MULTI-PLAYER MATRIX GAMES

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Game theory has had remarkable success as a framework for the discussion of animal behaviour and evolution. It suggested new interpretations and prompted new observational studies. Most of this work has been done with 2-player games. That is the individuals of a population compete in pairwise interactions. While this is often the case in nature, it is not exclusively so. Here we introduce a class of models for situations in which more than two (possibly very many) individuals compete simultaneously. It is shown that the solutions (i.e. the behaviour which may be expected to be observable for long periods) are more complex than for 2-player games. The concluding section lists some of the new phenomena which can occur. © 1997 Society for Mathematical Biology

1. Introduction. The modelling of animal behaviour by means of game theory provided a rich source of ideas to both biology and mathematics. It was used in studying very specific examples "in the field" and provided insights into the evolution of organisms as well as interesting mathematical concepts. Most noteworthy and fundamental of the latter is the evolutionarily stable strategy, which was coined by Maynard Smith and Price (1973). Almost all of this work was in the context of 2-player games. This is somewhat surprising because game theory has its origins in economics (specifically with von Neumann and Morgenstern 1944) in which multi-player models have always featured very strongly. Important texts in this area are Harsanyi and Selten (1988) and Binmore (1992). There is a very good reason why mathematical biologists have generally avoided multi-player games; their complexity. But now much is known about 2-player games, even when the interaction is in time and space, so that it is appropriate to start on their systematic study. The comparison and contrasting properties of the two types of games will be the basis of this paper.

It is to be expected that multi-player games will become increasingly studied if only because many naturally occurring situations can only be considered in this context, for example, hierarchies of animals or birds nesting in a colony. Even if games are pairwise there may be a structure to the collection of games which mean that the games cannot be considered in isolation and so are, in effect, a multi-player game. An extreme form of multi-player games is what Maynard Smith (1982) refers to as "playing the field." In this situation each individual is in competition against the whole population. However, the conflicts considered are usually somewhat indirect, e.g. establishing the best sex-ratio or gamete size.

In this paper we shall only consider matrix games which are *symmetric*. This means, *inter alia*, that the payoffs do not depend upon the ordering of the individuals which come together to play the game. Also the various opponents are always a random selection of the population as a whole. In section 5, the games will be further restricted to those in which all the players (in any particular contesting group) receive the same payoffs. We coin the term *super-symmetry* to describe this situation.

2. The Model.

2.1. *Evolutionarily stable strategies.* The concept of an evolutionarily stable strategy was introduced by Maynard Smith and Price (1973). Consider a population of animals competing for some resource such as food or mates. Individuals compete in (usually) pairwise games for a reward. Classically, it is assumed that all the members of the population are indistinguishable and that each individual is equally likely to face any other individual. We shall always assume that only a finite number of strategies is available to the players, these are the *pure strategies* which are labelled $1, \ldots, n$ and we let $\mathbf{U} = \{1, \ldots, n\}$ be this set of pure strategies.

Given the strategies played the outcome is determined; if player 1 plays i against player 2 playing j, then 1 receives the payoff or reward a_{ij} (2 receives a_{ji}) representing an adjustment in Darwinian fitness. The value a_{ij} can be thought of as an element in the $n \times n$ matrix $\mathbf{A}$, the *payoff matrix*. In this paper the matrix $\mathbf{A}$ is a constant; it does not depend upon the frequency with which the various strategies are being played. This is the usual assumption for matrix games.

An animal need not play the same, pure strategy every time. Instead it may play a *mixed strategy* i.e. play i with probability p_i for each of $i = 1, \ldots, n$. This implies that the strategy played by an animal is represented by a probability vector $\mathbf{p}$. With the assumption that the two protagonists are randomly chosen from the population, it follows that the expected payoff to a player playing $\mathbf{p}$ against an opponent playing $\mathbf{q}$, which is written as $E[\mathbf{p},\mathbf{q}]$, is

$$E[\mathbf{p},\mathbf{q}] = \sum p_i a_{ij} q_j = \mathbf{p}^T\mathbf{A}\mathbf{q}.$$

A fundamental question is, what strategies are likely to be prevalent in the population?

Suppose that $\mathbf{p}$ is played by almost all members of the population, the remainder of the population being a small, mutant group (constituting a fraction ε of the total population) playing $\mathbf{q}$. We say that $\mathbf{p}$ is *evolutionarily stable* (ES) against $\mathbf{q}$ if

$$E[\mathbf{p},(1-\varepsilon)\mathbf{p}+\varepsilon\mathbf{q}] > E[\mathbf{q},(1-\varepsilon)\mathbf{p}+\varepsilon\mathbf{q}],$$

for all, sufficiently small, ε. This implies that either

(i) $E[\mathbf{p},\mathbf{p}] > E[\mathbf{q},\mathbf{p}]$, or
(ii) $E[\mathbf{p},\mathbf{p}] = E[\mathbf{q},\mathbf{p}]$ and $E[\mathbf{p},\mathbf{q}] > E[\mathbf{q},\mathbf{q}]$.

If almost all players play $\mathbf{p}$, then almost all potential opponents are $\mathbf{p}$ players, so if $\mathbf{p}$ does better against $\mathbf{p}$ than $\mathbf{q}$ does, then $\mathbf{q}$ players will die out through natural selection. However, if they do equally well against $\mathbf{p}$, then how well the strategies perform against $\mathbf{q}$ becomes important. Consequently, in this case, $\mathbf{p}$ must then do better against $\mathbf{q}$ than $\mathbf{q}$ does for $\mathbf{p}$ to be ES against $\mathbf{q}$.

The vector $\mathbf{p}$ is said to be an evolutionarily stable strategy (ESS) if $\mathbf{p}$ is ES against all $\mathbf{q} \neq \mathbf{p}$. This means that if all members of a population play $\mathbf{p}$, then $\mathbf{p}$ cannot be invaded by a small group playing a different strategy. Thus the strategy $\mathbf{p}$ persists as the dominant strategy through time.

An alternative interpretation of ESS theory is to suppose that each individual plays a pure strategy and that it is the different frequencies of the various type of individuals which make up the ESS vector $\mathbf{p}$. The two approaches are indistinguishable for our purposes (except when we explicitly consider the time evolution in section 5.2) and we shall use the terminology appropriate to the situation in which all individuals are identical and each plays the ESS.

2.2. *Patterns of ESSs.*

Definition 2.2.1. Suppose that $\mathbf{p} = (p_i)$ is an ESS of the payoff matrix $\mathbf{A}$. The *support* of $\mathbf{p}$, $S(\mathbf{p})$, is the set $S(\mathbf{p}) = \{i\colon p_i > 0\}$. Thus it is the set of pure strategies that can be played by a $\mathbf{p}$-player.

Definition 2.2.2. Any collection of supports (with no repeated elements) is called a *pattern*. A particular pattern is the *pattern of the matrix* **A** if it is the collection of supports of the ESSs of **A**. A pattern is said to be *attainable* if there is some matrix **A** which has that pattern.

If a constant is added to all the entries in a column of the payoff matrix, then the set of ESSs is unaltered. Thus the pattern of $\mathbf{B} = (a_{ij} + c_j)$ is the same as that of **A**. In particular, with $c_j = -a_{jj}$ $\forall j$, the set of ESS of a matrix is the same as that of its *reduced matrix* (adding a constant to each column to make the leading diagonal terms zero). This useful result is due to Zeeman (1980).

The biological relevance of the concept of a pattern is as follows; if the same type of conflict is taking place in several locations, different ESSs may be observed. It is of interest to determine whether this difference in behaviour is due to differences in the local conditions (resulting in a different payoff matrix) or to the fact that the same payoff matrix may have more than one ESS. In the latter case the pattern of the matrix will consist of (at least) two entries. It is known that even low-order matrices may have very complicated patterns e.g. Cannings and Vickers (1988), Vickers and Cannings (1988a), Cannings and Vickers (1991).

A fundamental restriction on the pattern of a matrix is provided by the following theorem.

THE BISHOP–CANNINGS THEOREM 2.2.3. *If* $\mathbf{p}$ *is an ESS with support* I *and* $\mathbf{r} \neq \mathbf{p}$ *is an ESS with support* J, *then* $\mathrm{I} \not\supseteq \mathrm{J}$.

The original theorem, which is a little more general, appears in Bishop and Cannings (1976), where a proof is given. It follows immediately that if $I = \mathbf{U}$ then the ESS is unique.

2.3. *Multi-player games.* Much work has been done on 2-player games e.g. Hofbauer and Sigmund (1988), Maynard Smith (1982), Haigh (1975) and Cressman (1992). However, in the biological context, there has been little published work on multi-player games, examples are Haigh and Cannings (1989), Cannings and Whittaker (1995), and Broom, Cannings and Vickers (1996). This is due to both the wide applicability of 2-player models and to their relative simplicity.

There is a logical extension of the definition of an ESS from the 2-player case to the multi-player case. We will assume that the game is symmetrical among all the players in the sense that the order of the players is irrelevant (as well as the set of options available being the same for all players). Hence we can assume that any player that we are considering is the first player without loss of generality. We use the notation $E[\mathbf{p}; \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1}]$ to denote the expected payoff to an individual playing $\mathbf{p}$ in a conflict involving a total of n players, the strategies of its opponents being $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-1}$. When some of the strategies are the same, then we may write, for example, $E[\mathbf{p}; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3^{n-3}]$. As mentioned in the introduction, the set of opponents will be chosen at random from the complete population. It follows that the payoff is linear in each of the $\mathbf{p}$'s and so

$$E\left[\mathbf{p}; \{(1-\varepsilon)\mathbf{p}_1 + \varepsilon\mathbf{p}_2)\}^{n-1}\right] = \sum_{k=0}^{n-1} \binom{n-1}{k} (1-\varepsilon)^k \varepsilon^{n-k-1} E\left[\mathbf{p}; \mathbf{p}_1^k, \mathbf{p}_2^{n-k-1}\right].$$

A strategy $\mathbf{p}$ is ES against a strategy $\mathbf{q}$ if

$$\begin{aligned} &E[\mathbf{p}; (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}, (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}, \ldots, (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] \\ &\qquad > E[\mathbf{q}; (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}, (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}, \ldots, (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}], \end{aligned}$$

for all, sufficiently small ε. This condition is equivalent to the following;

A strategy $\mathbf{p}$ is ES against a strategy $\mathbf{q}$ if

either

$$E\left[\mathbf{p}; \mathbf{p}^{n-1}\right] > E\left[\mathbf{q}; \mathbf{p}^{n-1}\right],$$

or

$$E\left[\mathbf{p}; \mathbf{q}, \mathbf{p}^{n-2}\right] > E\left[\mathbf{q}; \mathbf{q}, \mathbf{p}^{n-2}\right] \quad \text{and} \quad E\left[\mathbf{p}; \mathbf{p}^{n-1}\right] = E\left[\mathbf{q}; \mathbf{p}^{n-1}\right],$$

or, there is a j $(2 \leqslant j \leqslant n-1)$ such that

$$E\left[\mathbf{p}; \mathbf{q}^j, \mathbf{p}^{n-j-1}\right] > E\left[\mathbf{q}; \mathbf{q}^j, \mathbf{p}^{n-j-1}\right]$$

and

$$E\left[\mathbf{p}; \mathbf{q}^i, \mathbf{p}^{n-i-1}\right] = E\left[\mathbf{q}; \mathbf{q}^i, \mathbf{p}^{n-i-1}\right] \qquad (0 \leqslant i \leqslant j-1).$$

More compactly, $\mathbf{p}$ is ES against $\mathbf{q}$ if there is a j $(0 \leqslant j \leqslant n-1)$ such that $E[\mathbf{p}; \mathbf{p}^{n-1-j}, \mathbf{q}^j] > E[\mathbf{q}; \mathbf{p}^{n-1-j}, \mathbf{q}^j]$ and $E[\mathbf{p}; \mathbf{p}^{n-1-i}, \mathbf{q}^i] = E[\mathbf{p}; \mathbf{p}^{n-1-i}, \mathbf{q}^i]$ for all n $i < j$. Naturally, $\mathbf{p}$ is an ESS if it satisfies a condition of this form for every $\mathbf{q} \neq \mathbf{p}$. An ESS which satisfies these conditions with j never more than J will be called be an ESS of level J. Note that for the generic case most of the preceding conditions will be superfluous (only ESSs of level 0 or 1 are required).

If $\mathbf{p}$ is an ESS then

$$E\left[\mathbf{p}; \mathbf{p}^{n-1}\right] \geqslant E\left[\mathbf{q}; \mathbf{p}^{n-1}\right] \qquad \forall \mathbf{q},$$

and since the payoffs are linear in each of the $\mathbf{p}$s it follows that

$$E\left[\mathbf{p}; \mathbf{p}^{n-1}\right] = E\left[\mathbf{q}; \mathbf{p}^{n-1}\right] \quad \text{whenever } S(\mathbf{q}) \subseteq S(\mathbf{p}).$$

3. Comparison with 2-Player Games. It is well known that for two players no two ESSs can have the same support. This is a consequence of Theorem 2.2.3. It is for this reason that the notion of a *pattern* described in the introduction is both extremely useful and relatively simple. However, for multi-player games the situation is very different. It is shown in section 4, while considering the possible collections of ESSs for 2-strategy, 3-player games, there can be ESSs with supports (1) and (1, 2), respectively, for the same payoff matrix. Thus the Bishop–Cannings theorem does not hold for three or more players. For four or more players it is even possible to have more than one internal ESS (i.e. an ESS in which each option is represented, so $p_i > 0\ \forall_i$), again see section 4.

THEOREM 3.1. *It is not possible to have two ESSs with the same support in a 3-player game.*

Proof. Suppose that $\mathbf{p}$ is an ESS of a 3-player game. Then one of the following three conditions holds for any $\mathbf{q} \neq \mathbf{p}$,

(i) $E[\mathbf{p}; \mathbf{p}, \mathbf{p}] > E[\mathbf{q}; \mathbf{p}, \mathbf{p}]$,
(ii) $E[\mathbf{p}; \mathbf{p}, \mathbf{p}] = E[\mathbf{q}; \mathbf{p}, \mathbf{p}]$ and $E[\mathbf{p}; \mathbf{q}, \mathbf{p}] > E[\mathbf{q}; \mathbf{q}, \mathbf{p}]$,
(iii) $E[\mathbf{p}; \mathbf{p}, \mathbf{p}] = E[\mathbf{q}; \mathbf{p}, \mathbf{p}]$, $E[\mathbf{p}; \mathbf{p}, \mathbf{q}] = E[\mathbf{q}; \mathbf{p}, \mathbf{q}]$ and $E[\mathbf{p}; \mathbf{q}, \mathbf{q}] > E[\mathbf{q}; \mathbf{q}, \mathbf{q}]$.

If $\mathbf{q}$ is a different ESS with the same support as $\mathbf{p}$, then

$$E[\mathbf{p}; \mathbf{p}, \mathbf{p}] = E[\mathbf{q}; \mathbf{p}, \mathbf{p}] \quad \text{and} \quad E[\mathbf{q}; \mathbf{q}; \mathbf{q}] = E[\mathbf{p}; \mathbf{q}, \mathbf{q}].$$

Thus condition (ii) is the only possibility and so

$$E[\mathbf{p}; \mathbf{q}, \mathbf{p}] > E[\mathbf{q}; \mathbf{q}, \mathbf{p}].$$

But concentrating upon $\mathbf{q}$ rather than $\mathbf{p}$ shows that

$$E[\mathbf{q}; \mathbf{p}, \mathbf{q}] > E[\mathbf{p}; \mathbf{p}, \mathbf{q}].$$

This contradiction completes the proof.

To summarise:

if $n = 2$ then Bishop–Cannings holds,
if $n = 3$ then Bishop–Cannings does not hold, but there cannot be more than one ESS with the same support.
if $n > 3$ then there can be more than one ESS with the same support.

4. 2-Strategy, *n*-Player Games. The case where there are only two strategies but there are any number of players is considered. This is the simplest non-trivial case and we find exactly what combinations of ESSs are possible. Label the strategies S_1 and S_2.

Define α_{ij} as the payoff to a player playing strategy i against $n-1$ players, j of which play strategy 2, i being equal to 1 or 2. Furthermore, define $\beta_j = \alpha_{1j} - \alpha_{2j}$.

4.1. *Pure ESSs.* If $\mathbf{p} = (1,0)$ represents the pure strategy S_1 then $E[\mathbf{p}; \mathbf{p}^{n-1}] = \alpha_{10}$. Similarly $E[\mathbf{q}; \mathbf{p}^{n-1}] = q_1\alpha_{10} + q_2\alpha_{20}$ where $\mathbf{q} = (q_1, q_2)$. The pure S_1 is thus an ESS of order 0 if $\alpha_{10} > \alpha_{20}$ i.e. if $\beta_0 > 0$. Similarly, the pure strategy S_2 is an ESS if $\alpha_{1n-1} < \alpha_{2n-1}$ or $\beta_{n-1} < 0$.

It follows in a similar manner for the non-generic case that S_1 is an ESS of order j if $\beta_j > 0$ and $\beta_i = 0$ $\forall i < j$ and that S_2 is an ESS of order j if $\beta_{n-1-j} < 0$ and $\beta_{n-1-i} = 0$ $\forall i < j$.

4.2. *Mixed ESSs.* The payoff to an individual playing $\mathbf{r} = (r_1, r_2)$ against a set of opponents, each of which is playing $\mathbf{p} = (p_1, p_2) = (p_1, 1 - p_1)$, is given by

$$\begin{aligned} E[\mathbf{r}; \mathbf{p}^{n-1}] &= E\left[\mathbf{r}; (p_1, p_2)^{n-1}\right] \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^{n-k-1} p_2^k E\left[\mathbf{r}; (1,0)^{n-k-1}, (0,1)^k\right] \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^{n-k-1} p_2^k (r_1\alpha_{1k} + r_2\alpha_{2k}). \end{aligned}$$

It follows that

$$\begin{aligned} &E[\mathbf{p}; \mathbf{p}^{n-1}] - E[\mathbf{q}; \mathbf{p}^{n-1}] \\ &\quad = \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^{n-k-1} p_2^k (p_1\alpha_{1k} + p_2\alpha_{2k} - q_1\alpha_{1k} - q_2\alpha_{2k}) \\ &\quad = \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^{n-k-1} p_2^k (p_1 - q_1)\beta_k \\ &\quad = (p_1 - q_1) p_2^{n-1} h(t), \end{aligned}$$

where

$$t = \frac{p_1}{1 - p_1} \quad \text{and} \quad h(t) = \sum_{k=0}^{n-1} \binom{n-1}{k} \beta_k t^{n-k-1}.$$

For $\mathbf{p}$ to be a level 0 ESS this expression must be positive whenever $\mathbf{q} \neq \mathbf{p}$ which is clearly impossible. In addition, if the expression is ever negative, then the corresponding value of $\mathbf{q}$ gives a strategy which invades $\mathbf{p}$. Hence the expression must be zero, so $h(p_1/p_2)=0$ is the equation that any ESS must satisfy.

Now consider $E[\mathbf{p};\mathbf{q},\mathbf{p}^{n-2}]-E[\mathbf{q};\mathbf{q},\mathbf{p}^{n-2}]$. Assuming w.l.o.g. that the $\mathbf{q}$ player is second in the order, we have

$$\begin{aligned}E[\mathbf{r};\mathbf{q},\mathbf{p}^{n-2}] &= r_1q_1\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\alpha_{1k}\\ &\quad+r_2q_1\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\alpha_{2k}\\ &\quad+r_1q_2\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\alpha_{1k+1}\\ &\quad+r_2q_2\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\alpha_{2k+1},\end{aligned}$$

which implies that

$$\begin{aligned}&E[\mathbf{p},\mathbf{q},\mathbf{p}^{n-2}]-E[\mathbf{q};\mathbf{q},\mathbf{p}^{n-2}]\\ &\quad=\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k(p_1q_1\alpha_{1k}+p_1q_2\alpha_{1k+1}+p_2q_1\alpha_{2k}+p_2q_2\alpha_{2k+1}\\ &\quad-q_1^2\alpha_{1k}-q_1q_2\alpha_{1k+1}-q_2q_1\alpha_{2k}-q_2^2\alpha_{2k+1})\\ &\quad=(p_1-q_1)\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k(q_1\beta_k+q_2\beta_{k+1}).\end{aligned}$$

Let

$$T_1=\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\beta_k \quad\text{and}\quad T_2=\sum_{k=0}^{n-2}\binom{n-2}{k}p_1^{n-k-2}p_2^k\beta_{k+1}.$$

Then

$$\begin{aligned}p_1T_1+p_2T_2&=\sum_{k=0}^{n-1}\binom{n-1}{k}p_1^{n-k-1}p_2^k\beta_k\\ &=p_2^{n-1}h(p_1/p_2)\\ &=0,\end{aligned}$$

at an ESS and so

$$
\begin{aligned}
E[\mathbf{p};\mathbf{q},\mathbf{p}^{n-2}] - E[\mathbf{q};\mathbf{q},\mathbf{p}^{n-2}] &= (p_1 - q_1)(q_1T_1 + q_2T_2) \\
&= (p_1 - q_1)[(q_1 - p_1)T_1 + (q_2 - p_2)T_2] \\
&= (p_1 - q_1)^2(T_2 - T_1) \\
&= -(p_1 - q_1)^2 T_1/p_2.
\end{aligned}
$$

Also

$$h'(t) = (n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}t^{n-k-2}\beta_k.$$

Hence, for a level 1 ESS it is required that T_1 be negative, or equivalently, that $h'(p_1/p_2)$ be negative. In summary, the conditions for a level 1 ESS at $\mathbf{p}$ are

(i) $h(t) = 0$,
(ii) $h'(t) < 0$,

where $t = p_1/(1 - p_1)$ and

$$h(t) = \sum_{k=0}^{n-1}\binom{n-1}{k}\beta_k t^{n-k-1}.$$

It can be shown that the conditions for an ESS of level j are

$$h(t) = 0, \qquad \frac{d^k h(t)}{dt^k} = 0 \qquad (k < j), \qquad \frac{d^j h(t)}{dt^j} < 0,$$

if j is odd. If j is even no such ESS can exist. Equivalently, $\mathbf{p}$ is an ESS of order j if $t_1 = p_1/(1 - p_1)$ is a root of order j of the polynomial $h(t)$ and the jth derivative at $t = t_1$ is negative.

Excluding polynomials which have double roots, ESSs correspond to the alternate roots of $h(t) = 0$ (if the polynomial has a negative derivative at one root then the derivative must be positive at the next and vice versa) so that the maximum number of internal ESSs is the integer part of $n/2$ (the polynomial is of order $n - 1$). In addition if there is a pure S_1 ESS then $h(1)$ is positive so that the last root does not correspond to an ESS, if there is a pure S_2 ESS then $h(0)$ is negative so that the first root does not correspond to an ESS. Hence the possible sets of ESSs are as follows:

0 pures, l internals l less than or equal to the integer part of $n/2$.
1 pure, l internals l less than or equal to the integer part of $n/2 - 1$.
2 pures, l internals l less than or equal to the integer part of $n/2 - 2$.

Note that the αs are unrestricted, so that any combination of ESSs (i.e. any values of the roots $t_1, \ldots, t_l$ of $h(t)=0$) is possible provided that the total number is in accordance with the preceding text.

5. 3-Player, 3-Strategy Games. After 2-strategy games the next simplest example is clearly that of 3-strategies. However, whereas it is possible to show exactly which collections of ESSs are attainable (not just collections of supports) when there are 2-strategies and n-players, this is very difficult to do for three strategies even for the 3-player case. Since for four or more players there can be more than one ESS per support, the notion of a pattern loses its attraction; however, for three players it is still useful. We shall consider which patterns are attainable for the general 3-player, 3-strategy case before considering some special cases.

The payoff to each member of a competing group is, from now on, assumed to be the same. They share the spoils equally. For a 2-player game, this results in a symmetric matrix. The adjective *super-symmetric* is descriptive of the situation that we have in mind. It follows that the payoffs are a_{ijk}, where each of i, j and k is 1, 2 or 3, and that the value of a_{ijk} is unchanged by a permutation of its subscripts. Thus, for example, a_{112} is the payoff to each member of a group of three players, two of which are playing strategy 1 and the third playing strategy 2.

A biological situation in which an equal division of the payoff may be appropriate is provided by a group of male frogs which are calling to attract mates. An individual might "cheat" by not calling but under favourable circumstances the payoff to that individual is primarily determined by the total number of callers. Also if a hunting animal joins a pack then its payoff is mainly determined by the success of the pack.

Consider a 3-player 3-strategy game with a population playing strategy $\mathbf{p} \in \mathbb{R}^3$. The mean fitness of the population is W, where

$$W = \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} a_{ijk} p_i p_j p_k.$$

The mean fitness is thus a homogeneous, cubic polynomial defined on the unit simplex, which here is a triangle (see Edwards (1977) for a discussion of the use of homogeneous coordinates in genetics). Also,

$$E[\mathbf{e}_1; \mathbf{p}^2] = E\left[\mathbf{e}_1; (p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3)^2\right]$$
$$= \sum_{j=1}^{3} \sum_{i=1}^{3} E[\mathbf{e}_1; \mathbf{e}_i, \mathbf{e}_j] p_i p_j$$

$$= \sum_{j=1}^{3} \sum_{i=1}^{3} a_{1ij} p_i p_j$$

$$= \frac{1}{3} \frac{\partial W}{\partial p_1},$$

where $\mathbf{e}_1$ is the unit vector corresponding to the pure strategy of always playing option 1, S_1. For an n-player, m-strategy game the general result is

$$E\left[\mathbf{e}_i; \mathbf{p}^{n-1}\right] = \frac{1}{n} \frac{\partial W}{\partial p_i} \qquad (1 \leqslant i \leqslant m).$$

In this case $\mathbf{p} \in \mathbb{R}^m$ and W is homogeneous of degree n.

Returning to the case $m = n = 3$, each pure strategy, S_i, corresponds to a vertex of the triangle of reference, which we choose to be equilateral and of unit height. The mixed strategy with $\mathbf{p} = (p_1, p_2, p_3)$ is a point of this triangle whose perpendicular distance from the edge opposite the vertex labelled i is p_i. There is a 1-1 correspondence between points of the triangle and possible strategies which the population could play. The mean fitness of a population playing a particular strategy is the value of W at the point representing that strategy.

If $\mathbf{p}^*$ is an ESS then

$$E\left[\mathbf{p}^*; \hat{\mathbf{p}}^{n-1}\right] > E\left[\mathbf{p}; \hat{\mathbf{p}}^{n-1}\right] \ \forall \hat{\mathbf{p}} = (1-\varepsilon)\mathbf{p}^* + \varepsilon \mathbf{p}$$

$$(\mathbf{p} \neq \mathbf{p}^* \text{ and } \varepsilon \text{ small and positive}).$$

Thus,

$$\frac{1}{n} \sum_i p_i^* \left.\frac{\partial W}{\partial p_i}\right|_{\hat{\mathbf{p}}} > \frac{1}{n} \sum_i p_i \left.\frac{\partial W}{\partial p_i}\right|_{\hat{\mathbf{p}}} \Rightarrow \sum_i (p_i^* - \hat{p}_i) \left.\frac{\partial W}{\partial p_i}\right|_{\hat{\mathbf{p}}} > 0,$$

from which it follows that W has a (local) maximum at $\mathbf{p}^*$. Thus ESSs are represented by local maxima within the bounds of the triangle.

5.1. *Attainable patterns for the* 3*-player case*. Theorem 3.1 shows that for three players it is not possible to have two ESSs with the same support. Also, in section 4 it was shown that it is not possible to have two pure ESSs as well as an internal ESS for the 2-strategy, 3-player case. If a pattern (with union V, $|V| = N$) is attainable as a set of ESSs on $n > N$ strategies, then it is also attainable as a set of ESSs on N strategies. Thus the pattern (1, 2) (1) (2), which is not attainable for two strategies, is not attainable for three strategies and neither is any pattern which contains this set of supports. Of the 2^7 patterns (there are seven non-empty supports each of which may or may not be in the pattern) there are only 40 permutationally distinct ones, and this restriction implies that 12 of these are unattainable.

The attainability of the 40 permutationally distinct patterns is given in Table 1, together with how to construct them if they are attainable. A pattern is labelled "special" if it attainable by a game with the form

$$a_{111} = a_{222} = a_{33} = 0 \quad \text{and} \quad a_{ijk} = \pm 1 \text{ otherwise}.$$

The table shows that no fewer than 17 of the patterns are attainable by games of this type. Such matrices were considered for the 2-player case in Cannings and Vickers (1988) as the result of a mapping of the ESSs to the cliques of a related graph. There are 32 such permutationally distinct sets of parameters and they are listed in Table 2 together with their pattern. The corresponding fitness surfaces are shown in Fig. 1. These diagrams will be discussed further in the next section when evolution is considered.

If a pattern is attainable on a set of strategies it is certainly attainable upon any superset of that set. Hence, since we know that (1); (1) (2); (1,2); (1), (1,2); are attainable for two strategies, they are for three, giving four more patterns. Such a pattern is labeled "subspace" in Table 1. This leaves seven to be decided. Four patterns were found using computer "trial and error" and are labelled "example" in Table 1. The pattern (1,2) (1,3) (2) (3) + I was found by construction (the space where this pattern exists is so small that the random search failed to find it).

Table 1. The complete list of 40 patterns for 3-player, 3-strategy, super-symmetric games and their attainability. An interior ESS, i.e. one with support (1, 2, 3), is denoted by I.

Pattern	Attainability	Pattern	Attainability
(1)	subspace	(1) + I	special
(1) (2)	subspace	(1) (2) + I	special
(1) (2) (3)	special	(1) (2) (3) + I	special
(1, 2)	subspace	(1, 2) + I	special
(1, 2) (1)	subspace	(1, 2) (1) + I	special
(1, 2) (3)	special	(1, 2) (3) + I	example (*a*)
(1, 2) (1) (2)	unattainable	(1, 2) (1) (2) + I	unattainable
(1, 2) (1) (3)	special	(1, 2) (1) (3) + I	example (*b*)
(1, 2) (1) (2) (3)	unattainable	(1, 2) (1) (2) (3) + I	unattainable
(1, 2) (1, 3)	special	(1, 2) (1, 3) + I	example (*c*)
(1, 2) (1, 3) (1)	special	(1, 2) (1, 3) (1) + I	special
(1, 2) (1, 3) (2)	special	(1, 2) (1, 3) (2) + I	example (*d*)
(1, 2) (1, 3) (1) (2)	unattainable	(1, 2) (1, 3) (1) (2) + I	unattainable
(1, 2) (1, 3) (2) (3)	special	(1, 2) (1, 3) (2) (3) + I	example (*e*)
(1, 2) (1, 3) (1) (2) (3)	unattainable	(1, 2) (1, 3) (1) (2) (3) + I	unattainable
(1, 2) (1, 3) (2, 3)	special	(1, 2) (1, 3) (2, 3) + I	special
(1, 2) (1, 3) (2, 3) (1)	special	(1, 2) (1, 3) (2, 3) (1) + I	unknown
(1, 2) (1, 3) (2, 3) (1) (2)	unattainable	(1, 2) (1, 3) (2, 3) (1) (2) + I	unattainable
(1, 2) (1, 3) (2, 3) (1) (2) (3)	unattainable	(1, 2) (1, 3) (2, 3) (1) (2) (3) + I	unattainable
ϕ	unattainable	I	special

Table 2. The 32 cases for the special class of super-symmetric games and their patterns. The coding corresponds to Fig. 1; + indicating that the matrix is as in the figure and − that all signs are changed.

a_{112}	a_{113}	a_{221}	a_{223}	a_{331}	a_{332}	a_{123}	Pattern	Figure code
−1	−1	1	−1	1	1	1	(1), (1, 2) (1, 2, 3)	1(a) +
1	1	−1	1	−1	−1	−1	(3) (1, 3) (2, 3) (1, 2, 3)	1(a) −
−1	−1	−1	−1	−1	−1	1	(1) (2) (3) (1, 2, 3)	1(b) +
1	1	1	1	1	1	−1	(1, 2) (1, 3) (2, 3)	1(b) −
−1	1	1	−1	−1	1	1	(1, 2) (1, 3) (2, 3) (1, 2, 3)	1(c) +
1	−1	−1	1	1	−1	−1	(1, 2) (1, 3) (2, 3)	1(c) −
1	1	−1	−1	−1	−1	1	(2) (3) (1, 2, 3)	1(d) +
−1	−1	1	1	1	1	−1	(1) (1, 2) (1, 3) (2, 3)	1(d) −
−1	−1	1	1	1	1	1	(1) (1, 2, 3)	1(e) +
1	1	−1	−1	−1	−1	−1	(2) (3) (1, 2) (1, 3)	1(e) −
−1	1	1	1	1	1	1	(1, 3) (1, 2, 3)	1(f) +
1	−1	−1	−1	−1	−1	−1	(2) (3) (1, 2)	1(f) −
1	−1	−1	−1	−1	1	1	(2) (1, 2) (2, 3)	1(g) +
−1	1	1	1	1	−1	−1	(1, 3) (1, 2, 3)	1(g) −
−1	1	1	1	1	−1	1	(1, 3) (1, 2, 3)	1(h) +
1	−1	−1	−1	−1	1	−1	(2) (1, 2) (2, 3)	1(h) −
1	−1	−1	−1	−1	−1	1	(2) (3) (1, 2)	1(i) +
−1	1	1	1	1	1	−1	(1, 3) (2, 3)	1(i) −
1	−1	−1	−1	1	−1	1	(2) (1, 2) (1, 3)	1(j) +
−1	1	1	1	−1	1	−1	(1, 3) (2, 3)	1(j) −
1	−1	1	−1	−1	−1	1	(3) (1, 2)	1(k) +
−1	1	−1	1	1	1	−1	(1, 3) (2, 3)	1(k) −
−1	1	−1	1	−1	1	1	(1, 3) (2, 3)	1(l) +
1	−1	1	−1	1	−1	−1	(1, 2) (1, 3)	1(l) −
−1	−1	−1	1	1	1	1	(1) (2, 3)	1(m) +
1	1	1	−1	−1	−1	−1	(3) (1, 2) (1, 3)	1(m) −
−1	1	−1	1	1	1	1	(1, 3) (2, 3)	1(n) +
1	−1	1	−1	−1	−1	−1	(3) (1, 2)	1(n) −
−1	1	1	1	−1	1	1	(1, 3) (2, 3)	1(o) +
1	−1	−1	−1	1	−1	−1	(2) (1, 2) (1, 3)	1(o) −
1	1	1	1	1	1	1	(1, 2, 3)	1(p) +
−1	−1	−1	−1	−1	−1	−1	(1) (2) (3)	1(p) −

It is impossible to have no ESS in the super-symmetric case since the point in the triangle with the maximum payoff is an ESS (except for degenerate cases where there is an infinite number of points which achieve the maximum). However, this pattern is attainable in general 3-player, 3-strategy games.

The attainability of one pattern, *viz.* (1, 2) (1, 3) (2, 3) (1) + I, is unknown.

5.2. *Dynamics*. The previous sections of this paper have dealt with multi-player games in a static sense. We shall now consider these games dynamically, modelling the way the strategy mix in the population changes through time with the assumption that only pure strategies exist. There are two types of dynamic, the discrete dynamic and the continuous dynamic. The discrete dynamic models the frequency of each pure strategy in the

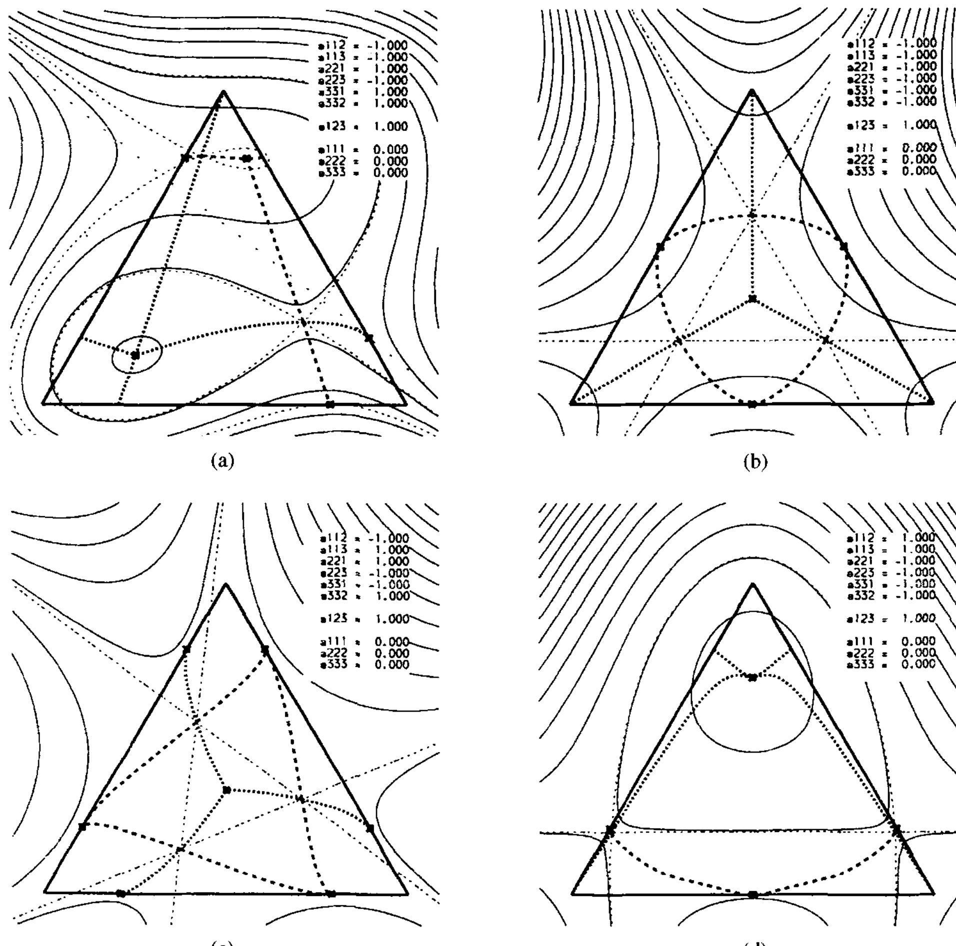


Figure 1. The fitness and dynamics of the 16 permutationally distinct cases with $a_{111} = a_{222} = a_{333} = 0$ and $a_{ijk} = \pm 1$ otherwise. The bold triangle is the triangle of reference. The thin, continuous curves are the contours of mean fitness, the thin dotted lines are the special contours through the equilibrium points and the thin dashed (straight) line is an asymptote of the contours. The thick dotted lines are the out sets of the saddle points and the thick dashed lines are the in sets. Nodes (other than vertices) are starred. Every ESS is a node and a node is an ESS of $\mathbf{A}$ or $-\mathbf{A}$. The in sets and out sets delineate the basins of attraction.

population in successive generations. The continuous dynamic models the same process but now time is a continuous variable. This latter model corresponds to overlapping generations.

The discrete dynamic relates the current strategy $\mathbf{p}$ to that in the next generation, $\mathbf{p}'$, by the equation (for an n-player, m-strategy game)

$$p_i' = p_i \frac{\left(E\left[\mathbf{e}_i; \mathbf{p}^{n-1}\right] + C\right)}{E\left[\mathbf{p}; \mathbf{p}^{n-1}\right] + C} \qquad (1 \leqslant i \leqslant m),$$

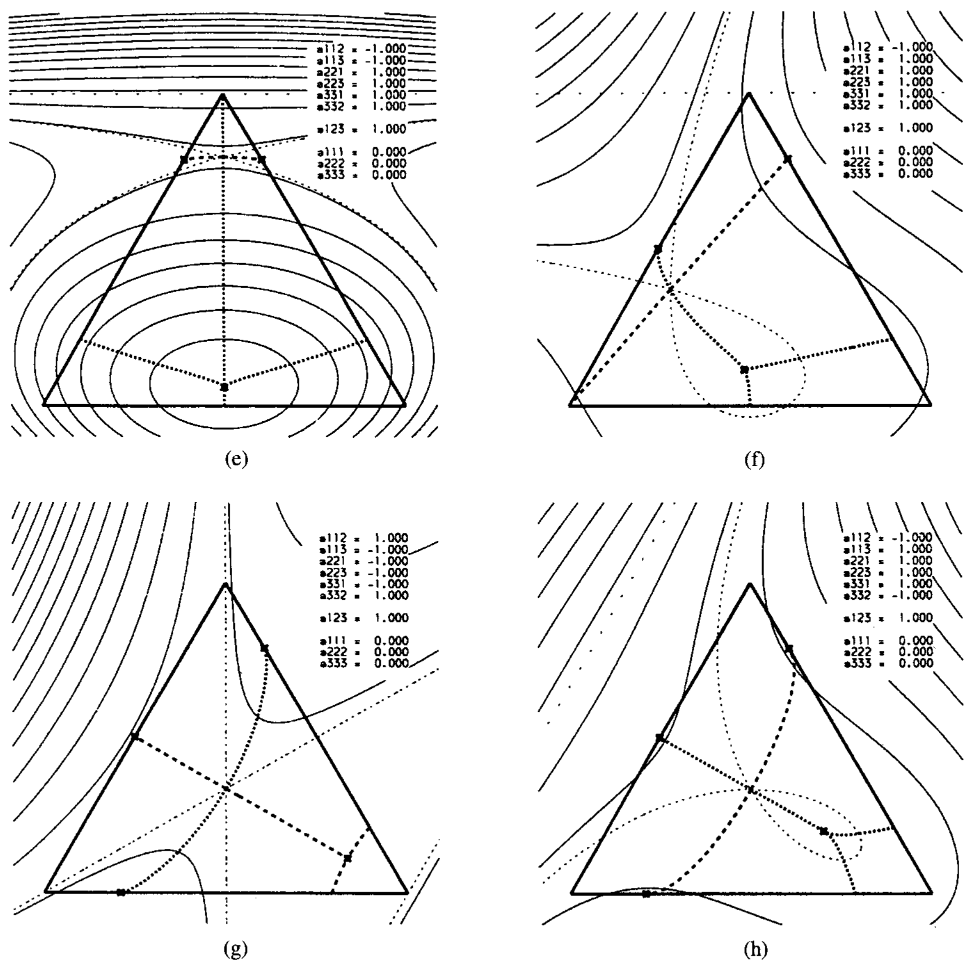


Figure 1. (*Continued*).

or, equivalently,

$$p_i' = p_i\left(\frac{1}{n}\frac{\partial W}{\partial p_i} + C\right)\Bigg/(W+C) \qquad (1 \leqslant i \leqslant m).$$

The constant C is needed to ensure that all of the new frequencies are positive, or, more biologically, to ensure that the fitness of every individual is non-negative. Baum and Eagon (1967) show that, for a set of recurrence equations of this form, the mean fitness, W, increases monotonically. Now the continuous dynamic is given by the following system of differential equations,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = p_i\left(\frac{1}{n}\frac{\partial W}{\partial p_i} - W\right), \qquad (1 \leqslant i \leqslant m),$$

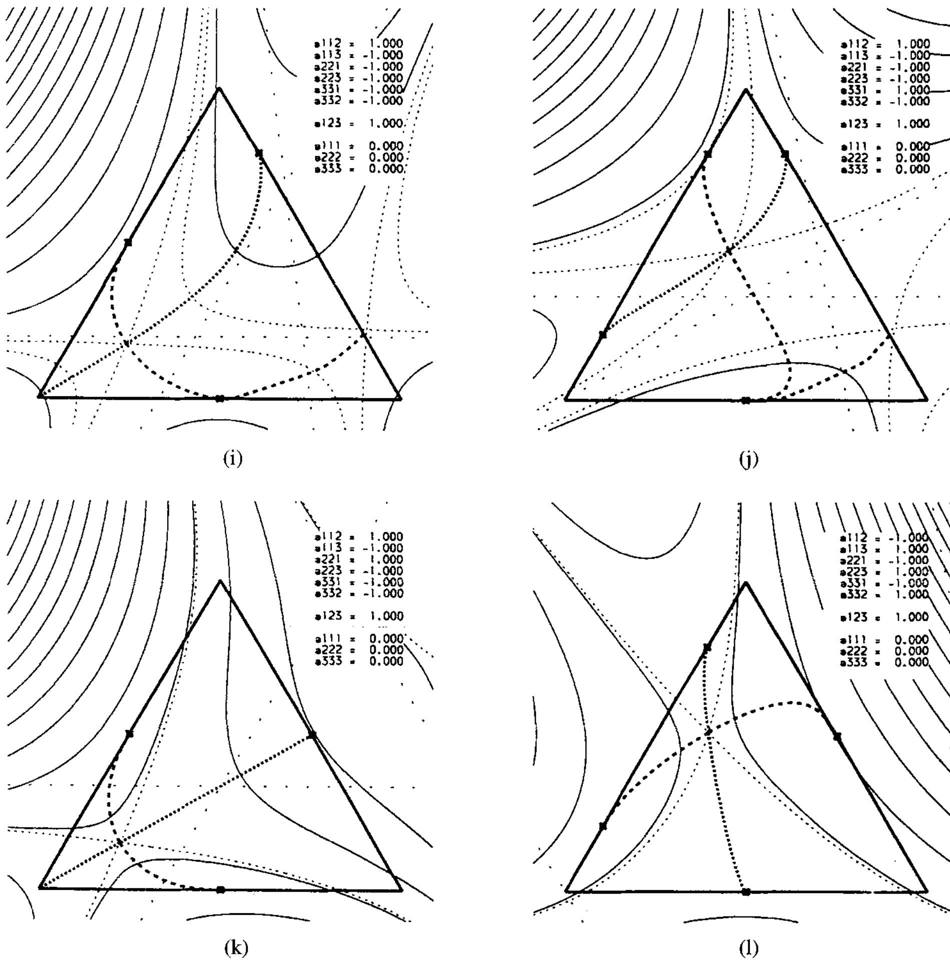


Figure 1. (*Continued*).

and here also W increases monotonically. Thus there is no qualitative difference between the two types of dynamics. In each case the system will converge to a maximum of W, which corresponds to an ESS. Note that if a constant is added to every element of a payoff matrix, the ESSs are unchanged. Unusually, here the discrete dynamic also does not change the stability of equilibrium points. Evolution happens more slowly the larger the constant added to all the terms, as the constant tends to infinity the discrete dynamic tends to the continuous dynamic (whose behaviour is unaffected by the addition of a constant). Only the continuous dynamic will be explicitly considered here and furthermore we mainly consider the special games in which

$$a_{111} = a_{222} = a_{333} = 0 \quad \text{and} \quad a_{ijk} = \pm 1 \text{ otherwise}.$$

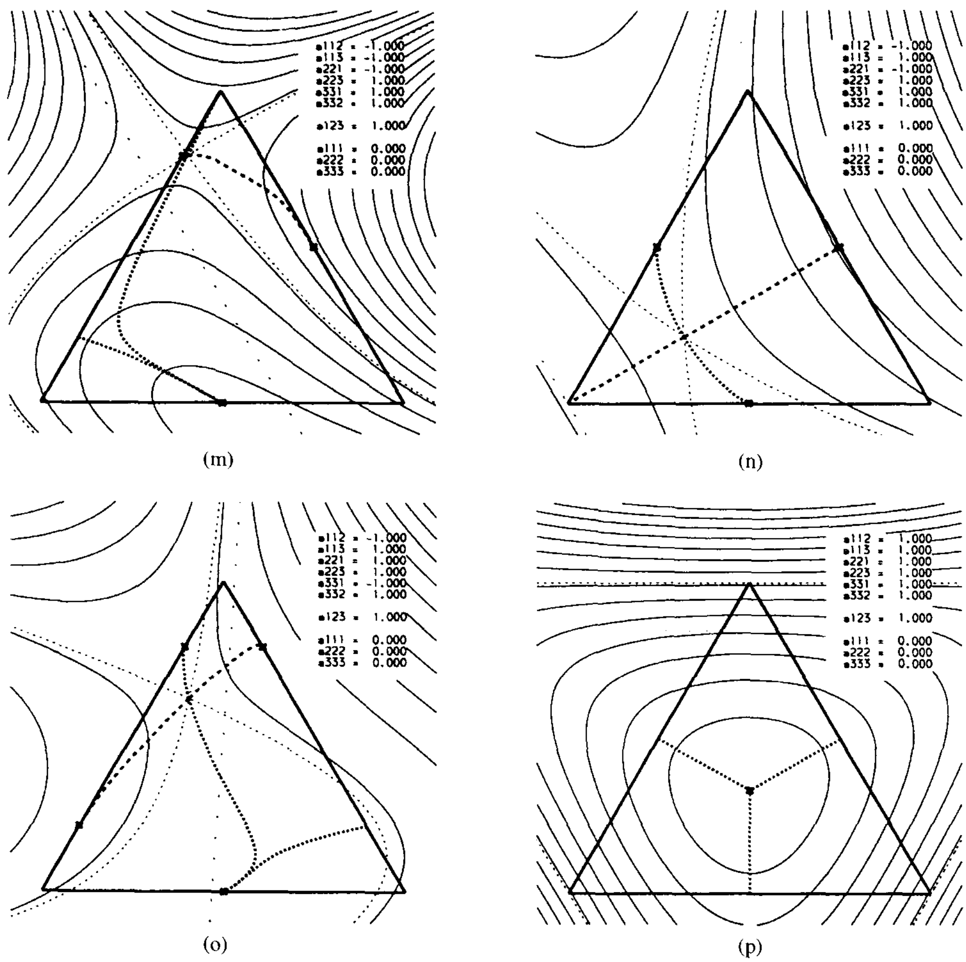


Figure 1. (*Continued*).

The corresponding problems for the 2-player, 3-strategy situation are to have

$$\text{(i)} \qquad \mathbf{A} = \pm\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \text{(ii)}\ \mathbf{A} = \pm\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In case (i) the trajectories are given by

$$(p_1 - p_3)^3 - p_2(p_1^2 - p_3^2) + 2p_1p_2p_3\ln(p_1/p_3) = Cp_1p_2p_3,$$

where C is any constant. Taking the positive sign for $\mathbf{A}$, there are two ESSs $(1/2, 1/2, 0)$ and $(0, 1/2, 1/2)$ whose basins of attraction are separated by the line $p_1 = p_3$. For the negative sign, the ESSs are now $(0,1,0)$, $(1/2,0,1/2)$ and they are separated by

$$(p_1 - p_3)^3 - p_2(p_1^2 - p_3^2) + 2p_1 p_2 p_3 \ln(p_1/p_3) = 0.$$

In case (ii) the trajectories are curves of the form

$$\sum_{i=1}^{3} \frac{\alpha_i}{p_1} = 0 \quad \text{where} \sum_{i=1}^{3} \alpha_i = 0.$$

The positive sign for **A** gives the unique ESS $(1/3,1/3,1/3)$ and the negative sign gives the three pure strategies as ESSs, their basins of attraction being separated by the lines $p_2 = p_3$, $p_1 = p_3$ and $p_1 = p_2$. We see that even in this very special case the analytic solutions are not particularly simple. Indeed, we were unable to persuade Maple to provide any solutions to the first case. Not surprisingly, therefore, in the following presentation of the solutions for the 3-player case, we concentrate upon the qualitative behaviour.

Figure 1 shows the contours of the mean fitness for each of the special games and also the basins of attraction of each of the ESSs (for the continuous dynamic). The boundaries of these basins are the in and out sets of the unstable equilibrium points (saddles). All nodes (other than those which happen to be pure strategies) are marked by a star. An ESS is always a node and a node is always an ESS for either **A** or $-$**A**. Each of the 16 diagrams gives two patterns since each gives information for both **A** and $-$**A**. The last column in Table 2 indicates which of the 16 diagrams is involved and also whether the pattern refers to **A** (shown by +) or $-$**A** (shown by $-$). There are 17 different patterns including some with four ESSs (e.g. Fig. 1(a) with $-$**A** and 1(b) with **A**) thus these very special payoffs cover a wide range of possibilities (though not as great a proportion as the equivalent special case for two players, where all attainable 3-strategy patterns and all but one attainable 4-strategy patterns are achieved by such matrices). Fig. 1(a) demonstrates clearly how Bishop–Cannings is violated for 3-player games (in total 18 out of the 32 cases violate Bishop–Cannings).

Some of the diagrams have several ESSs but with one of them having a very large basin of attraction, so that it is likely that this will be the ESS that the population tends to, e.g. Fig. 1(c) with $-$**A**, although if pure strategies are introduced sequentially and the population allowed to settle to an ESS each time, then it is possible to reach any of the ESSs (by suitably ordering the introduction of the strategies). In Figs. 1(b) and 1(c) (each with $-$**A**), a sequential introduction would prevent the population reaching the mixed ESS even though it has the largest basin of attraction. Note that for symmetric, 2-player games it is always possible to reach an

ESS if the strategies are introduced in a particular order. A complete discussion of this is to be found in Cannings, Tyrer and Vickers (1993).

Figure 2 shows the trajectories for the five examples referred to in Table 1. These provide patterns which are not attainable by the special payoffs considered in Fig. 1. Finally, Fig. 3 shows a set of payoffs which give another result which is not possible for the case of two players. For symmetric matrices (equivalent to the genetic problem of multiple alleles at one locus) if new alleles–strategies are introduced into the population sequentially and the population allowed to converge to an ESS between each introduction, then if a new allele–strategy invades the current ESS it must feature in the support of the new ESS (see Vickers and Cannings 1988b). This is not true for the game in Fig. 3. If we start with the strategy 1

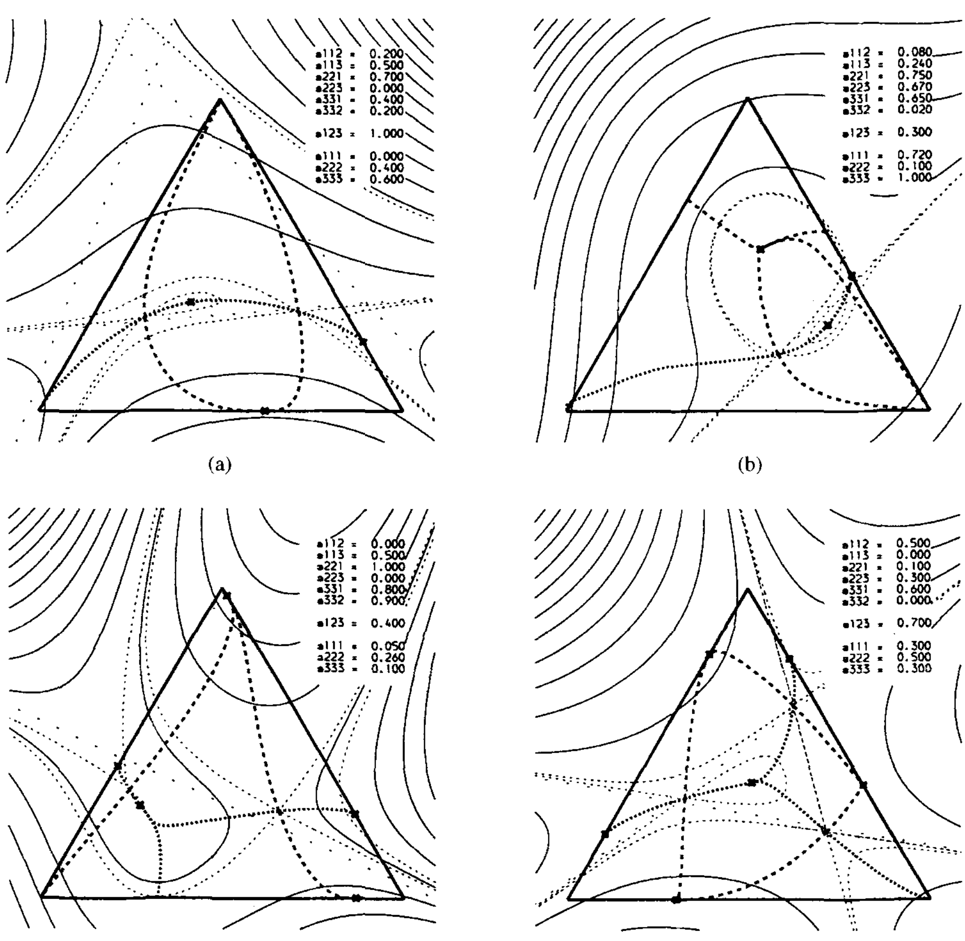
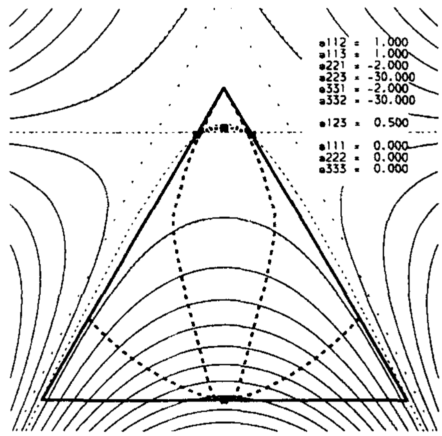


Figure 2. The fitness and dynamics of 5 patterns which are not attainable by the special payoffs of Fig. 1. The coding of the lines is the same as in Fig. 1. They are in alphabetical order as shown in Table 1.



(e)

Figure 2. (*Continued*).

and then introduce strategy 2 the population converges to a mixture of the two strategies. If now the third strategy is introduced then the population follows the out set of the equilibrium point indicated in Fig. 3 and converges to the pure ESS, strategy 2. Thus the first strategy is eliminated by the introduction of the third, but this third strategy does not feature in the final population, i.e. it acts as a catalyst.

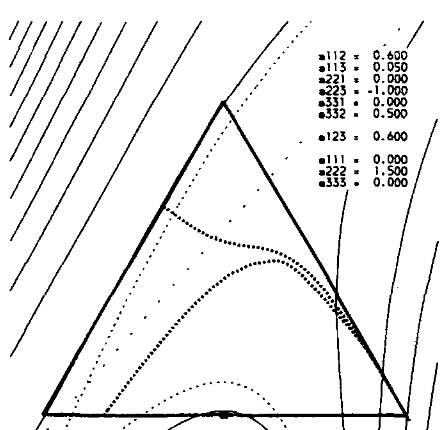


Figure 3. This demonstrates that even though a new strategy may be able to invade what was an ESS, it need not be represented in the final ESS. Specifically, strategy 3 can invade $(1, 2)$ but the outcome is (2).

6. Conclusion. Multi-player games are common and were much observed in field studies. One type of example is provided by leks. Males group together and females visit the lek solely for mating, the males providing no parental care. Clutton-Brock et al. (1988) lists 7 species of mammal and Oring (1982) 35 species of bird in which lekking occurs. The strategy of a male involves deciding when to arrive at the lek, how long to remain and how much energy to expend in displaying. Communal nests are another example. Female ostriches and also female groove billed anis (Krebs and Davies 1981) have communal nests (although the details are quite different). It is the timing of the egg laying which is particularly important.

In this paper we are not attempting to model any specific biological situation. Rather a framework is presented which is sufficiently flexible to cover many real-world problems. Our aims have been to show multi-players games can be modelled and to demonstrate that the structure of the solutions (even in the special, super-symmetric contests) is considerably more complicated than that for classical, 2-player games. Even for 3-player, 3-strategy matrix games (with the super-symmetry condition) there are new features:

(i) the support of one ESS may be contained in another,
(ii) there are restrictions on the patterns attainable, for example, $\{(1), (1,2), (2)\}$ is impossible,
(iii) if a new strategy appears (through mutation, perhaps) which can invade, then it may cause the system to move to a new ESS in which it does not feature,
(iv) an ESS (with a large basin of attraction) need not be attainable by a sequential introduction of strategies.

With 4-player games, there may even be ESSs with identical supports. One of the fundamental problems of applying game theory to economic models is that of deciding which of several equilibria (often Nash equilibria) is the "best." This is the problem addressed by Harsanyi and Selten (1988). The same problem was studied in the biological context, mainly by way of travelling waves, but perhaps it has not received the attention that it deserves. It is clear that as the study of multi-player, multi-strategy games progresses, it will be a serious challenge to relate theory and observation.

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