MA3615 Groups and Symmetry

Solutions to Exercise Sheet 6

1. (a) C_3 (same as rotational symmetries of a triangle).
   (b) D_8 (same as all symmetries of a square).
   (c) D_4 (same as all symmetries of a rectangle).
   (d) D_4 (as (c)).
   (e) $\{e\}$.
   (f) C_4 (same as rotational symmetries of a square).
   (g) D_6 (same as all symmetries of a triangle).
   (h) D_2.

2. First determine $|G|$. Look at the action of G on the set of white vertices. Each vertex can be mapped to any other by a rotation of the cube. So if v is any white vertex then $|Orb_G(v)| = 4$. The stabilizer of v, G_v, consists of all rotations around the main diagonal through the vertex v, so $|G_v| = 3$. Hence using the orbit stabilizer theorem we get that
$$|G| = |Orb_G(v)|.|G_v| = 4.3 = 12.$$
We now construct an explicit isomorphism with A_4. Number the white vertices 1,2,3,4. The action of G on the set of white vertices gives a homomorphism
$$\psi : G \to S_4.$$
What is Kerψ? It's easy to see that if $g \in G$ fixes all the white vertices then it must fix all the black vertices as well, so we must have $g = e$. Thus we have Ker$\psi = \{e\}$.

   What is Imψ? G contains two types of rotations: rotations around a main diagonal of the cube and rotations around an axis through the middle of opposite faces. A rotation around a main diagonal corresponds, under ψ, to a permutation with cycle type of the form $(a)(b, c, d)$. A rotation around an axis through the middle of opposite faces corresponds, under ψ, to a permutation with cycle type $(a, b)(c, d)$. Both types of permutations lie in A_4. So we have in fact that
$$\psi : G \to A_4.$$
As Ker$\psi = \{e\}$, this map is one-to-one and as $|G| = |A_4| = 12$, it must be onto as well, so it is an isomorphism and we have $G \cong A_4$ as required.

3. Let G be the rotational symmetry group of this solid. Consider the action of G on the two tetrahedrons ($|X| = 2$). It is easy to see that there is a rotation which sends one tetrahedron to the other. So if T is one of the tetrahedron, then $|Orb_G(T)| = 2$. What is the stabilizer of T in G, G_T? This is the rotational symmetry group of T, namely A_4. So we have $|G_T| = |A_4| = 12$. Thus using the Orbit-Stabilizer theorem we see that
$$|G| = |Orb_G(T)||G_T| = 2.12 = 24.$$

Using the classification of finite rotation groups we have that

$$G \cong C_{24}, D_{24} \quad \text{or} \quad S_4.$$

Now, G is not abelian, as G has a subgroup isomorphic to A_4 (which is not abelian). So G cannot be isomorphic to C_{24}. Moreover, G does not contain a rotation of order 12. So G cannot be isomorphic to D_{24}, which has an element of order 12 (namely r). Hence $G \cong S_4$.

4. Let G be the rotational symmetry group of the cuboctahedron. Consider the action of G on the set X consisting of all square faces ($|X| = 6$). It is easy to see that there is always a rotation which sends a given square face to any other. So if F denotes a given square face then we have $|Orb_G(F)| = 6$. What is G_F? G_F consists of the rotations around the axis passing through the middle of F. So $|G_F| = 4$. Thus using the orbit-stabilizer theorem we get that

$$|G| = |Orb_G(F)|.|G_F| = 6.4 = 24.$$

Using the classification we get $G \cong C_{24}, D_{24}$ or S_4. As G is not abelian it cannot be isomorphic to C_{24} and as it does not contain a rotation of order 12, it cannot be isomorphic to D_{24}. Hence we must have $G \cong S_4$ as required.