Calculus: Questions 1
Partial Differentiation

The first six questions are from last year's sheet 7 and are included here for revision. The other questions are new.

1. Find *all* the first and second partial dervatives of

   (a) $6x^2 + 4y(1-x) + (1-y)^2$ (b) $\sin(x^2 y)$

2. If $f(x,y) = x^2y^2$ with $x = \cos t$ and $y = \sin t$, find $\frac{df}{dt}$ and $\frac{d^2f}{dt^2}$ by *using partial differentiation.*

3. Using partial differentiation find $\frac{dy}{dx}$ where

   (a) $(x-1)y^3 + x^2\cos x = 3$ (b) $\cos(xy) = 0$

4. Find *all* the stationary points (i.e. points where $f_x = f_y = 0$) of

   $$f(x,y) = e^{x+y}(x^2 + y^2 - xy),$$

   and find their natures (i.e. are they maxima, minima or saddle points?)

5. Find all the stationary points of the function

   $$f(x,y) = (x+y)^4 - x^2 - y^2 - 6xy$$

   and identify their type.

6. Show that the function

   $$f(x,y) = x^2y^2 - 2xy(x+y) + 4xy$$

   has stationary points at $(1,1)$ and $(2,0)$. Find the three other stationary points.

   Identify the type of *all* the stationary points of this function.

7. **From lectures** Find the stationary points of

   $$f(x,y) = 2x^4 + 8x^2y^2 - 4(x^2 - y^2) + 2$$

   Identify the type of *all* the stationary points of this function.

8. Find the stationary points of

   $$f(x,y) = -x^2 - y^3 + 12y^2$$

   Identify the type of *all* the stationary points of this function.

9. How would you solve question 6 if you were told that you had to maximise $f(x,y)$ with the constraint $x^2 + y^2 = 4$?

Solutions

1. (a) $f_x = 12x - 4y,\ f_y = 4(1-x) - 2(1-y),\ f_{xx} = 12,\ f_{xy} = f_{yx} = -4,\ f_{yy} = 2.$

   (b) $f_x = 2xy\cos(x^2y),\ f_y = x^2\cos(x^2y),\ f_{xx} = 2y\cos(x^2y) - 4x^2y^2\sin(x^2y),$
   $f_{xy} = f_{yx} = 2x\cos(x^2y) - 2x^3y\sin(x^2y),\ f_{yy} = -x^4\sin(x^2y).$

2. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2xy^2\frac{dx}{dt} + 2x^2y\frac{dy}{dt}$

   $= 2\cos t\sin^2 t \times (-\sin t) + 2\cos^2 t\sin t \times (\cos t) = -2\sin^3 t\cos t + 2\sin t\cos^3 t$

   $\frac{d^2f}{dt^2} = \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2$

   $= 2xy^2\frac{d^2x}{dt^2} + 2x^2y\frac{d^2y}{dt^2} + 2y^2\left(\frac{dx}{dt}\right)^2 + 8xy\frac{dx}{dt}\frac{dy}{dt} + 2x^2\left(\frac{dy}{dt}\right)r$

   $= 2\cos t\sin^2 t\times(-\cos t) + 2\cos^2 t\sin t\times(-\sin t) + 2\sin^2 t\times(-\sin t)^2 + 8\cos t\sin t\times(-\sin t)\times(\cos t) + 2\cos^2 t\times(\cos t)^2 = 2\cos^4 t - 12\cos^2 t\sin^2 t + 2\sin^4 t.$

3. Using the result that if $f(x,y) = C$ then $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$. Note that it is often better to leave both x and y in your answers.

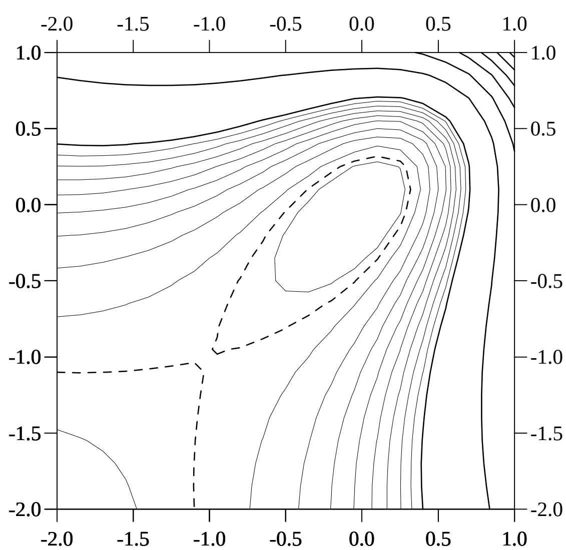
   (a) $y^3 + 2x\cos x - x^2\sin x + 3(x-1)y^2\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{y^3 + 2x\cos x - x^2\sin x}{3(x-1)y^2}$.

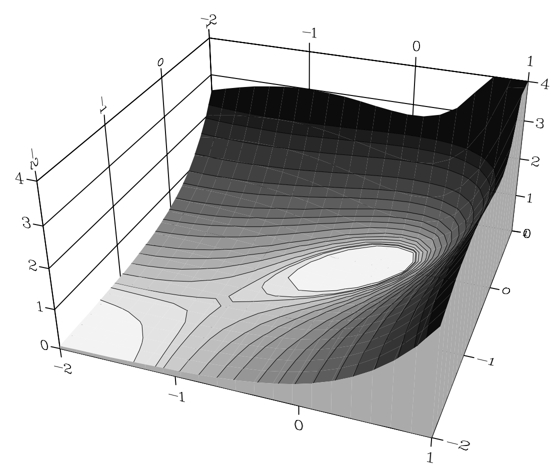
   (b) $-y\sin(xy) - x\sin(xy)\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -y/x$.

4. Stationary points at $(x,y) = (0,0)$ and $(-1,-1)$.

   $(0,0)$ is a **minimum**.

   $(-1,-1)$ are **saddle point**.

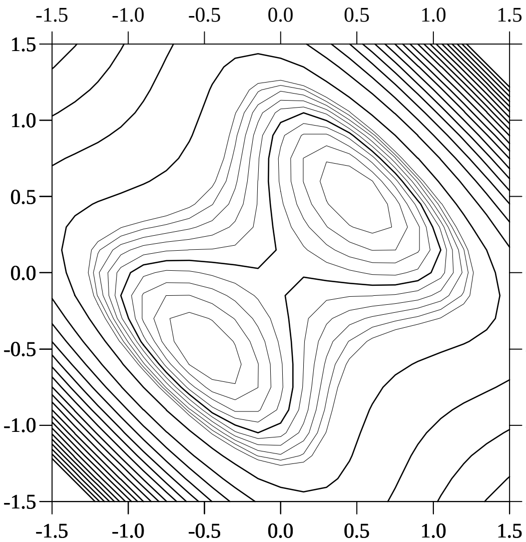


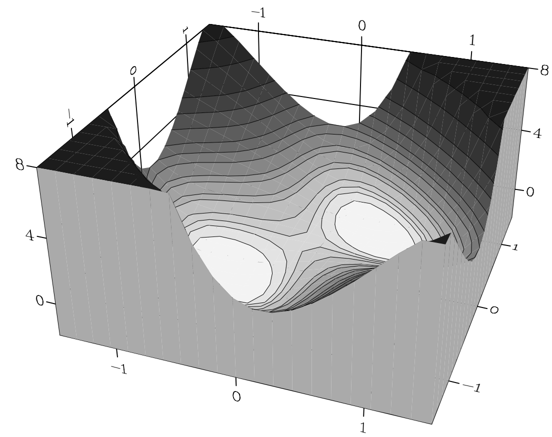


5. Stationary points at $(x, y) = (0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$.

   $(0, 0)$ is a **saddle point**.

   $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ are **minima**.

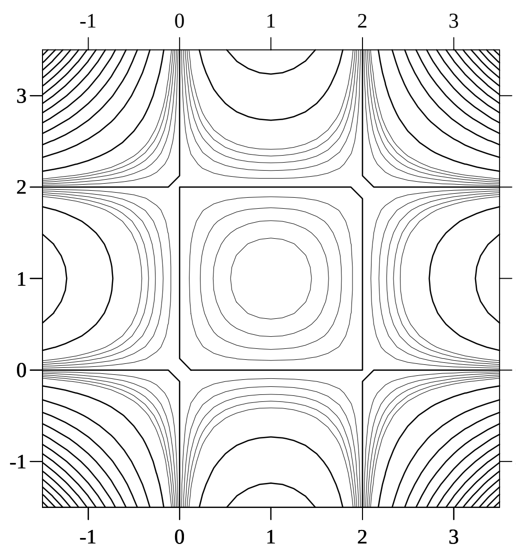


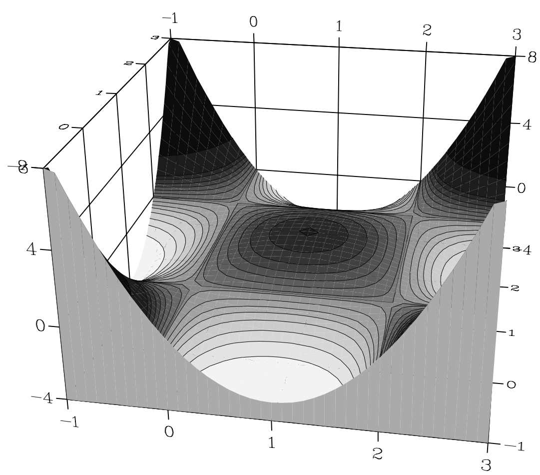


6. Stationary points at $(x, y) = (0, 0)$, $(1, 1)$, $(2, 2)$, $(2, 0)$ and $(0, 2)$.

   $(0, 0)$, $(2, 2)$, $(2, 0)$ and $(0, 2)$ are **saddle points**.

   $(1, 1)$ is a **maximum**.





7. Stationary points at $(x, y) = (0, 0)$, $(1, 0)$, $(-1, 0)$.

   $(0, 0)$ is a saddle point, the others are minima.

8. Stationary points at $(x, y) = (0, 0)$, $(0, 8)$.

   $(0, 0)$ is a saddle point, $(0, 8)$ is a maximum.

9. See lectures