Calculus: Questions 3
Integration with Two Variables

1. Find the following integrals

   (a) $\int_0^1 \int_0^1 xy^2 dx\, dy$

   (b) $\int_0^1 \int_0^x (x+y)^2 dy\, dx$

   (c) $\int\int_R xy\, dx\, dy$ where R is the rigion given by $0 \leq x$, $0 \leq y$ and $x + y \leq 1$.

   (d) $\int\int_R x + y\, dx\, dy$ where R is the region given by $0 \leq x \leq 2$, $0 \leq y \leq 2$ and $xy \leq 1$. [Hint you may find it easier to split your region of integration into smaller convenient areas.]

2. By changing the order of integration evaluate

$$\int_0^1 \int_{\tan^{-1} x}^{\pi/4} \frac{y^2}{\tan y} + \frac{x}{\sin y}\, dy\, dx$$

3. Sketch the region of integration in the x–y plane for the following integral

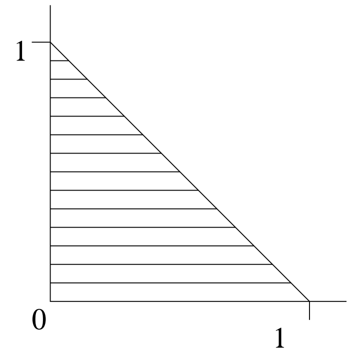
$$I_1 = \int_0^1 \int_{x^2}^1 \sqrt{y} \sin(x\sqrt{y})\, dy\, dx.$$

   Change the order of integration, and hence evaluate the integral.

1. (a) 1/6

   (b) 7/12

   (c) Region of integration is



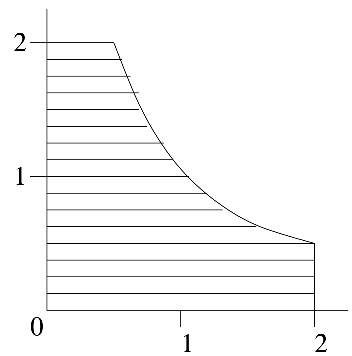
$$I = \int_0^1 \int_0^{1-y} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2y}{2}\right]_0^{1-y} dy = \int_0^1 \frac{(1-y)^2y}{2}\,dy = \frac{1}{2}\int_0^1 y - 2y^2 + y^3\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}$$

   (d) Region of integration is

Split integration into two regions, say above and below line $y = 1/2$.

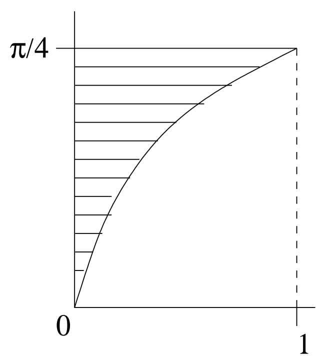


$$I = \int\int_R x+y\,dx\,dy = \int_{1/2}^2 \int_0^{1/y} x+y\,dx\,dy + \int_0^{1/2}\int_0^2 x+y\,dx\,dy$$

$$= \int_{1/2}^2 \left[x^2/2 + xy\right]_0^{1/y} dy + \int_0^{1/2} \left[x^2/2 + xy\right]_0^2 dy$$

$$= \int_{1/2}^2 \frac{1}{2y^2} + 1\,dy + \int_0^{1/2} 2 + 2y\,dy = \left[-\frac{1}{2y} + y\right]_{1/2}^2 + \left[2y + y^2\right]_0^{1/2} = \frac{7}{2}$$

2. Region of integration is

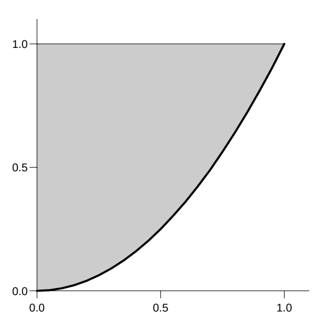


$$I = \int_0^{\pi/4} \int_0^{\tan y} \frac{y^2}{\tan y} + \frac{x}{\sin y}\,dx\,dy$$

$$= \int_0^{\pi/4} \left[\frac{xy^2}{\tan y} + \frac{x^2}{2\sin y}\right]_0^{\tan y} dy = \int_0^{\pi/4} \frac{y^2 \tan y}{\tan y} + \frac{\tan^2 y}{2\sin y}\,dy$$

$$= \int_0^{\pi/4} y^2 + \frac{\sin y}{2\cos^2 y}\,dy = \left[\frac{y^3}{3} + \frac{1}{2\cos y}\right]_0^{\pi/4} = \frac{\pi^3}{192} + \frac{1}{\sqrt{2}} - \frac{1}{2}$$

3. Region of integration is



$$I = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\sin(x\sqrt{y})\,dx\,dy$$

$$= \int_0^1 \left[-\cos(x\sqrt{y})\right]_0^{\sqrt{y}} dy = \int_0^1 -\cos(y) + 1\,dy$$

$$= \left[-\sin(y) + y\right]_0^1 = 1 - \sin 1.$$