Calculus: Questions 4
Change of Variable

1. An ellipse is given by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
You want to find the area by using a change of coordinates: $x = r\cos\theta$, $y = \frac{br}{a}\sin\theta$. Find the range of values of r and θ that correspond to the interior of the ellipse.

   Find the Jacobian of the transformation.

   Find the area.

2. Find the Jacobians of these variable changes

   (a) $x = a\cosh s\cos t$, $y = a\sinh s\sin t$ (elliptic coordinates)

   (b) $x = st$, $y = (t^2 - s^2)/2$ (parabolic coordinates)

3. The moment of inertia of a flat object in the x-y plane about the z axis is defined to be
$$I = \iint \rho(x, y)(x^2 + y^2)dx\,dy$$
where $\rho(x, y)$ is the mass per unit area of the object.

   Find the moment of inertia of a circular disc centred on the origin of radius a and mass per unit area ρ_0.

4. (a bit trickier?) repeat this exercise for the ellipse in question 1.

5. Spherical polar coordinates (see lectures) are defined by
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
Find the Jacobian of this transformation.

Solutions

1. $J = br/a$, Area $= \pi ab$
2. (a) $J = a^2(\sinh^2 s + \sin^2 t)$
   (b) $J = s^2 + t^2$
3. $I = \displaystyle\int_0^{2\pi} \int_0^a \rho_0 r^3 \, dr \, d\theta = \pi \rho_0 a^4/2$
4. $I = \pi \rho_0 ab(a^2 + b^2)/4$
5. $J = r^2 \sin\theta$