Calculus: Questions 5
Laplace Transforms

1. Find the Laplace transforms (i.e. not by looking up tables!) of
   (a) x
   (b) x^n
   (c) $\cosh ax$
   (d) $1 - \cos ax$

   Then check in the tables you got the right answer.

2. Use Laplace transforms to solve
   (a) $y'' + 4y' + 3y = 0$, with $y(0) = 0$ and $y'(0) = 2$.
   (b) $y'' + 2y' = x$, $y(0) = 1$ and $y'(0) = 0$.

3. Derive the result
$$\mathcal{L}(xf(x)) = -F'(p).$$

4. Derive the p-shifting theorem for Laplace transforms:
$$\mathcal{L}(e^{ax}f(x)) = F(p - a).$$

5. y(x) satisfies the differential equation
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = f(x),$$
   Show that the Laplace transform $Y(p) = \mathcal{L}(y(x))$ for general initial conditions $y(0)$ and $y'(0)$ is given by
$$Y(p) = \frac{y'(0) + (p + 2)y(0) + F(p)}{p^2 + 2p + 2}$$
   where $F(p) = \mathcal{L}(f(x))$ is the Laplace transform the function on the right of the above equation.

   If the initial coditions are $y(0) = y'(0) = 0$. If $f(t) = 2.$, find $F(p)$.

   Express $Y(p)$ in the following form:
$$Y(p) = \frac{A}{p} + \frac{B(p + 1)}{(p + 1)^2 + 1} + \frac{C}{(p + 1)^2 + 1},$$
   where A, B and C are constants to be determined. Use the reult of the previous question to find $y(x)$.

   Checkk that your solution satisfies the initial conditions $y(0) = y'(0) = 0$.

Solutions

1. (a) $1/p^2$
   (b) $n!/p^{n+1}$
   (c) $p/(p^2 - a^2)$
   (d) $\dfrac{a^2}{p(p^2 + a^2)}$

2. (a) $y = e^{-x} - e^{-3x}$
   (b) $y = 9/8 - x/4 + x^2/4 - e^{-2x}/8$

3. Consider
$$\frac{d}{dp}\int_0^\infty f(x)e^{-px}dx = \int_0^\infty \frac{\partial}{\partial p}\left(f(x)e^{-px}\right)dx = \int_0^\infty -xf(x)e^{-px}dx$$
so
$$\frac{d}{dp}F(p) = \mathcal{L}(-xf(x))$$
Hence result.

4.
$$\mathcal{L}(e^{ax}f(x)) = \int_0^\infty e^{ax}f(x)e^{-px}dx = \int_0^\infty f(x)e^{-(p-a)x}dx$$
This last term is just the definition of $F(p)$ but with p replaced by $p - a$. Hence result.

5. $F(p) = 2/p$
$$Y(p) = \frac{F(p)}{p^2 + 2p + 2} = \frac{2}{p(p^2 + 2p + 2)} = \frac{1}{p} - \frac{(p+1)}{(p+1)^2 + 1} - \frac{1}{(p+1)^2 + 1},$$
Hence
$$y(x) = 1 - e^{-x}\cos x - e^{-x}\sin x$$
Don't forget to do the check!