Calculus: Questions 6
More Laplace Transforms

1. Use Laplace transforms to solve

$$y'' + 4y' + 3y = r(x) \quad \text{with} \quad y(0) = y'(0) = 0,$$

and

$$r(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & 2 < x \end{cases}$$

2. The height of the mass suspended on a spring above its equilibrium position when its support is in its rest position is $x(t)$. The equation for the motion of the mass is

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 5y(t),$$

where $y(t)$ is the vertical displacement of the support above its rest position .

Show that the Laplace transform $X(p) = \mathcal{L}(x(t))$ of the vertical displaceme nt of the mass for general initial conditions $x(0)$ and $x'(0)$ is given by

$$X(p) = \frac{x'(0) + (p+4)x(0) + 5Y(p)}{p^2 + 4p + 5}$$

where $Y(p) = \mathcal{L}(y(t))$ is the Laplace transform of the vertical displaceme nt of the support.

The mass is initially at rest ($x(0) = x'(0) = 0$). After $t = 0$ the support is moved vertically upwards as

$$y(t) = t/5.$$

Find $Y(p)$ and hence $X(p)$

Express $X(p)$ in the following form:

$$X(p) = \frac{A}{p^2} + \frac{B}{p} + \frac{C(p+2)}{(p+2)^2+1} + \frac{D}{(p+2)^2+1},$$

where A, B, C and D are to be determined. Hence find the displacement of the mass $x(t)$. Check that your solution satisfies the initial conditions $x(0) = x'(0) = 0$. [Use transform tables]

3. Use the convolution integral to find the inverse Laplace transform of

(a)

$$Y(p) = \frac{1}{p^2(p^2+1)}$$

(b)

$$Y(p) = \frac{1}{(p^2+1)^2}$$

Solutions

1. Take LT to get

$$(p^2+4p+3)Y = \frac{1}{p} - \frac{e^{-2p}}{p}$$

$$Y = \frac{1}{p(p+1)(p+3)} - \frac{e^{-2p}}{p(p+1)(p+3)}$$

$$\frac{1}{p(p+1)(p+3)} = \frac{1/3}{p} - \frac{1/2}{p+1} + \frac{1/6}{p+3}$$

The inverse of this bit is

$$\mathcal{L}^{-1}\left(\frac{1}{p(p+1)(p+3)}\right) = \frac{1}{3} - \frac{e^{-x}}{2} + \frac{e^{-3x}}{6}$$

So

$$\mathcal{L}^{-1}\left(\frac{e^{-2p}}{p(p+1)(p+3)}\right) = \begin{cases} 0 & x<2 \\ \frac{1}{3} - \frac{e^{-(x-2)}}{2} + \frac{e^{-3(x-2)}}{6} & 2<x \end{cases}$$

Just subtract this from your prvious answer to get the solution.

2. $Y(p) = 1/5p^2$

$$X(p) = \frac{1}{p^2(p^2+4p+5)} = \frac{1/5}{p^2} + \frac{-4/25}{p} + \frac{(4/25)(p+2)}{(p+2)^2+1} + \frac{3/25}{(p+2)^2+1},$$

$$x(t) = \frac{t}{5} - \frac{4}{25} + \frac{4}{25}e^{-2t}\cos t + \frac{3}{25}e^{-2t}\sin t$$

Don't forget to check $x(0) = x'(0) = 0$!

3. (a)

$$Y(p) = \frac{1}{p^2} \times \frac{1}{p^2+1}$$

so

$$y(x) = \int_0^x (x-x')\sin x'\, dx' = [-(x-x')\cos x']_0^x - \int_0^x \cos x'\, dx' = x - [\sin x']_0^x = x - \sin x$$

(b)

$$Y(p) = \frac{1}{p^2+1} \times \frac{1}{p^2+1}$$

$$y(x) = \int_0^x \sin(x-x')\sin x'\, dx' = \int_0^x \frac{1}{2}\left(\cos((x-x')-x') - \cos((x-x')+x')\right) dx'$$

$$\int_0^x \frac{1}{2}\left(\cos(x-2x') - \cos(x)\right) dx' = = \frac{1}{2}\left[-\frac{1}{2}\sin(x-2x') - x'\cos x\right]_0^x$$

$$= \frac{\sin x}{4} - \frac{x\sin x}{2} + \frac{\sin x}{4} = \frac{\sin x}{2} - \frac{x \sin x}{2}$$