Calculus: Questions 8
Differential Equations 2

1. Find the Wronskian, $W(x)$, of the following pairs of functions

   (a) $y_1(x) = x^2 - 2x + 1, \quad y_2(x) = x^2 + 2x + 1.$

   (b) $y_1(x) = e^x, \quad y_2(x) = e^{2x}.$

   (c) $y_1(x) = e^x, \quad y_2(x) = xe^x.$

   (d) $y_1(x) = \sin x, \quad y_2(x) = \cos x.$

   (e) $y_1(x) = \sin^2 x, \quad y_2(x) = \cos^2 x.$

   (f) $y_1(x) = \sin^2 x, \quad y_2(x) = \cos^2 x \; - 1.$

2. Find the general solutions to the following homogeneous equations in the form

$$y = Ay_1(x) + By_2(x).$$

   In each case verify that the Wronskian of $y_1(x)$ and $y_2(x)$ is non-zero.

   (a) $\dfrac{d^2y}{dx^2} - y = 0$

   (b) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

   (c) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$

   (d) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0$

   (e) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

   (f) $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 0$

3. Solve the inhomogeneous equation

$$\frac{d^2y}{dx^2} - y = x$$

   using the techniques from last year.

   Now determine the particular integral using

$$y(x) = y_1(x)v_1(x) + y_2(x)v_2(x)$$

   where

$$v_1(x) = -\int \frac{y_2R(x)}{W(x)}\,dx \qquad v_2(x) = \int \frac{y_1R(x)}{W(x)}\,dx$$

   where $R(x)$ is the expression on the right hand side of the differential equation (i.e, $R(x) = x$ here). Check the two solutions are the same.

Solutions

1. Remember $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

   (a) $W(x) = -4(x^2 - 1)$
   (b) $W(x) = e^{3x}$
   (c) $W(x) = e^{2x}$
   (d) $W(x) = -1$
   (e) $W(x) = -2\sin x \cos x$
   (f) $W(x) = 0$

2. Note, if you get different y_1 and y_2 (say, you have them the other way round) you may get different Wronskians.

   (a) $y(x) = Ae^x + Be^{-x}, \quad W(x) = -2.$
   (b) $y(x) = Ae^{-x} + Be^{-2x}, \quad W(x) = -e^{-3x}.$
   (c) $y(x) = A + Be^x, \quad W(x) = e^x.$
   (d) $y(x) = Ae^{-4x}\sin 3x + Be^{-4x}\cos 3x, \quad W(x) = -3e^{-8x}.$
   (e) $y(x) = Ax + Bx^2, \quad W(x) = x^2$. Note: you can't define a solution by specifying y and y' at $x = 0$ where the Wronskian is zero.
   (f) $y(x) = \dfrac{A}{x} + \dfrac{B\ln x}{x}, \quad W(x) = 1/x^3$

3. Using undetermined constants, try $y(x) = ax + b$, finding $a = -1$, $b = 0$. General solution is

$$y = Ae^x + Be^{-x} - x.$$

   With $y_1(x) = e^x$ and $y_2(x) = e^{-x}$, $W(x) = -2$:

$$v_1 = -\int \frac{e^{-x}x}{-2}\,dx = -\frac{xe^{-x}}{2} - \frac{e^{-x}}{2}$$

$$v_2 = \int \frac{e^{x}x}{-2}\,dx = -\frac{xe^{x}}{2} + \frac{e^{+x}}{2}$$

$$y_1(x)v_1(x) + y_2(x)v_2(x) = -\frac{x}{2} - \frac{1}{2} - \frac{x}{2} + \frac{1}{2} = -x$$

   Particular integrals are the same!