Calculus: Questions 9
Differential Equations 3

1. Find the general solutions to the following inhomogeneous equations. In each case find the particular integrals using the method of variation of parameters.

   (a) $\dfrac{d^2y}{dx^2} - y = 1$

   (b) $\dfrac{d^2y}{dx^2} - y = e^x$

   (c) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^x$

   (d) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 1$

   (e) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3$

2. (From lectures) Solve the inhomogeneous equation

$$x\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2.$$

Solutions

1. In each case find solutions y_1 and y_2 to the homogeneous equation, and find the particular integral by using

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = -\int \frac{y_2(x)R(x)}{W(x)}\,dx, \qquad u_2(x) = \int \frac{y_1(x)R(x)}{W(x)}\,dx$$

where $R(x)$ is the right hand side of the differential equation in standard form, and $W(x) = y_1(x)y_2'(x) - y_1(x)y_2'(x)$.

   (a) $y_1 = e^x, \quad y_2 = e^{-x}, \quad W(x) = -2, \quad u_1(x) = -e^{-x}/2, \quad u_2(x) = -e^x/2, \quad y(x) = -1$

   (b) $y_1 = e^x, \quad y_2 = e^{-x}, \quad W(x) = -2, \quad u_1(x) = x/2, \quad u_2(x) = -e^{2x}/4,$
   $y(x) = xe^x/2 - e^x/4.$

   (c) $y_1 = e^{-x}, \quad y_2 = e^{-2x}, \quad W(x) = -e^{-3x}, \quad u_1(x) = e^{2x}/2, \quad u_2(x) = -e^{3x}/3,$
   $y(x) = e^x/3.$

   (d) $y_1 = 1, \quad y_2 = e^x, \quad W(x) = e^x, \quad u_1(x) = -x, \quad u_2(x) = -e^{-x}, \quad y(x) = -x - 1.$

   (e) $y_1 = x, \quad y_2 = x^2, \quad W(x) = x^2$
   Note: equation not in standard form — divide by x^2 to get

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2y}{x^2} = x$$

   $R(x) = x$, $u_1(x) = -x^2/2, \quad u_2(x) = x, \quad y(x) = x^3/2.$

2. You need to spot one solution to get started. The easy one (as spotted in lectures) is $y_1(x) = x$. Then do all the stuff from lectures and you get

$$y = Ax + Bx\int \frac{e^x}{x^2}\,dx - x^2 - 2x\ln x + 2.$$