AS2051

Section 1: Differentiation

Partial Derivatives

For a function $f(x_1, ..., x_n)$ of n variables and an integer k between 1 and n, the partial derivative

$$\frac{\partial f}{\partial x_k} = f_{x_k}$$

of f with respect to the variable x_k is the derivative of f with respect to x_k only, with the remaining $n-1$ variables are all held fixed.

Partial Derivatives

Formally the partial derivative of $f(x, y)$ with respect to x is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Example

Consider the function

$$f(x, y) = x^2 + 3xy + 4y^2$$

Evaluate f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx} and show that $f_{xy} = f_{yx}$

Solution

$$f_x = 2x + 3y$$

$$f_y = 3x + 8y$$

$$f_{xx} = 2$$

$$f_{yy} = 8$$

$$f_{xy} = 3$$

$$f_{yx} = 3$$

As we can see, $f_{xy} = 3 = f_{yx}$ as required.

In this last example we were considering a simple case where:

- x and y are not stated as been a function of a third variable t, say, which would make the function f also a function of t
- The two variables x and y are not such that one depends on the other e.g. y is not a function of x.

We will now consider briefly each of these two cases.

$f(x, y)$ where x and y are themselves functions of t

We can write f explicitly as $f(x(t), y(t))$ and so our function is a function of t. What is the derivative of f with respect to t?

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

Consider $f(x, y) = x^2 + 3xy + 4y^2$ where $x(t) = t$ and $y(t) = t^2$. Calculate

$$\frac{df}{dt}$$

.

Solution

We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Calculate each component:

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x + 8y$$

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = 2t$$

Substituting in to the equation:

$$\begin{aligned}\frac{df(x(t), y(t))}{dt} &= (2x + 3y)1 + (3x + 8y)2t \\ &= (2t + 3t^2) + 2t(3t + 8t^2) \\ &= 2t + 3t^2 + 6t^2 + 16t^3 \\ &= 2t + 9t^2 + 16t^3\end{aligned}$$

Note

If x and y are functions of two variables (t and s say) then:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$y(x)$ and $x(y)$

Note that in some cases $y(x)$ or $x(y)$. Therefore,

- If $y(x)$ then $f(x, y(x))$ so f is really a function of x only.
- If $x(y)$ then $f(x(y), y)$ so f is really a function of y only.

For the first of these two cases then:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Example

Consider

$$f(x, y(x)) = \tan^{-1}\left(\frac{x}{y(x)}\right)$$

where $y(x) = \sin(x)$. Calculate

$$\frac{df}{dx}$$

Solution

$$\begin{aligned}\frac{df(x, y(x))}{dx} &= \left(\frac{1}{1 + \left(\frac{x}{y}\right)^2}\right)\frac{1}{y} + \left(\frac{1}{1 + \left(\frac{x}{y}\right)^2}\right)\left(\frac{-x}{y^2}\right)\cos x \\ &= \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2}\cos x \\ &= \frac{\sin x}{x^2 + \sin^2 x} - \frac{x \cos x}{x^2 + \sin^2 x}\end{aligned}$$

Note

Note that for cases where $f(x, y(x))$, if $f(x, y) = \text{constant}$ then

$$\frac{df}{dx} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Classify stationary Points for a Function of 2 Variables

We first determine where both

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

Then we can go on to classify the stationary points.

- If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at a stationary point then the point is a minimum. (Note that these conditions imply that $f_{yy} > 0$)
- If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at a stationary point then the point is a maximum. (Note that these conditions imply that $f_{yy} < 0$)
- If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at a stationary point then the point is a saddle point.

Example

Consider

$$f(x,y) = 2x^4 + 8x^2y^2 - 4(x^2 - y^2) + 2$$

Find and classify the stationary points.

Solution

First we need to determine the points where

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= 8x^3 + 16xy^2 - 8x \\ &= 8x(x^2 + 2y^2 - 1) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= 16x^2y + 8y \\ &= 8y(2x^2 + 1) \end{aligned}$$

Therefore $8x(x^2 + 2y^2 - 1) = 0$ and $8y(2x^2 + 1) = 0$.
Considering the second equation $(2x^2 + 1) \neq 0$ and so $y = 0$.
Then $8x(x^2 + 2y^2 - 1) = 0$ gives $x = 0$ or $x = \pm 1$.

So then there are three stationary points that we need to classify $(0,0)$, $(1,0)$, $(-1,0)$. Now

$$\frac{\partial^2 f}{\partial x^2} = 24x^2 + 16y^2 - 8$$

$$\frac{\partial^2 f}{\partial y^2} = 16x^2 + 8$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 32xy$$

We can now determine the nature of each of the three stationary points:

x	y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	type
0	0	-8	8	0	-64	saddle point
1	0	16	24	0	384	minimum
-1	0	16	24	0	384	minimum

Maxima & Minima of Functions of more than 2 Variables

How do we determine maxima, minima for functions of more than two variables?

The necessary condition for extremum for a function (e.g. $f(x_1, x_2, x_3, ..., x_n)$ of n variables is:

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} \cdots \frac{\partial f}{\partial x_n} = 0$$

i.e. $f_{x_1} = f_{x_2} = \ldots\ldots f_{x_n} = 0$.

How can we tell if, the points satisfying this criterion are, for example, maxima?

We need to consider the Taylor series for n variables.

$$\begin{aligned} & f(x_1 + \delta x_1, x_2 + \delta x_2, ...x_n + \delta x_n) = f(x_1, x_2, ...x_n) \\ + & \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\ldots \frac{\partial f}{\partial x_n}\right) \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \vdots \\ \delta x_n \end{pmatrix} \\ + & \frac{1}{2}(\delta x_1, \delta x_2, \ldots\ldots \delta x_n) \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & f_{x_nx_n} \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \vdots \\ \delta x_n \end{pmatrix} \\ + & \ldots\ldots\ldots\ldots \end{aligned}$$

Given that we know we need

$$f_{x_1} = f_{x_2} = \ldots\ldots f_{x_n} = 0.$$

To determine the nature of the points we look at the next term in the Taylor series. This term must be positive for a minimum and negative for a maximum for all possible $(\delta x_1, \delta x_2, \delta x_3 \ldots\ldots \delta x_n)$.If a matrix is real and symmetric all the eigenvalues are positive for a min and negative for a max. A point that satisfies :

$$f_{x_1} = f_{x_2} = \ldots\ldots f_{x_n} = 0.$$

to be a minimum requires

$$|H_1| = |f_{xx}| > 0 \quad |H_2| = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix} > 0$$

$$|H_3| = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix} > 0$$

up to $|H_n| > 0$

For a maxima we need to have

$$|H_1| < 0 \quad |H_2| > 0 \quad |H_3| < 0 \quad |H_4| > 0$$

etc (so alternating $< >$ starting with $<$)

Example

Let

$$f(x,y,z) = x^2y^2z^2 - x^2 - y^2 - z^2$$

Find the stationary points and determine if maxima or minima (or can't tell from this test).

Solution

First we calculate all of the first derivatives.

$$\begin{aligned} f_x &= 2xy^2z^2 - 2x \\ &= 2x(y^2z^2 - 1) \end{aligned}$$

$$\begin{aligned} f_y &= 2yx^2z^2 - 2y \\ &= 2y(x^2z^2 - 1) \end{aligned}$$

$$\begin{aligned} f_z &= 2x^2y^2z - 2z \\ &= 2z(x^2y^2 - 1) \end{aligned}$$

For a stationary point we require $f_x = f_y = f_z = 0$.
So we need:

$$2x(y^2z^2 - 1) = 0 \quad (1)$$

$$2y(x^2z^2 - 1) = 0 \quad (2)$$

$$2z(x^2y^2 - 1) = 0 \quad (3)$$

Equation (1) implies either $x = 0$ or $y^2z^2 - 1 = 0$.Consider the first of these, $x = 0$ then substituting into equation (2) we deduce that $y = 0$. Then equation (3) implies that $z = 0$. So one stationary points is $(0, 0, 0)$.

Now we consider if $x \neq 0$, $y \neq 0$, $z \neq 0$. Then:

$$(y^2z^2 - 1) = 0 \quad (4)$$

$$(x^2z^2 - 1) = 0 \quad (5)$$

$$(x^2y^2 - 1) = 0 \quad (6)$$

From equations (4) and (5) we obtain

$$\begin{aligned} (y^2z^2 - 1) - (x^2z^2 - 1) &= 0 \\ z^2(y^2 - x^2) &= 0 \quad (7) \end{aligned}$$

Now,$z \neq 0$ and so $x = \pm y$. Substituting into equation (6) we get $x = \pm 1$. Then we can obtain stationary points at $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

Now to examine the stationary points. First we note that

$$H_1 = \left(2y^2z^2 - 2 \right)$$

$$H_2 = \begin{pmatrix} 2y^2z^2 - 2 & 4xyz^2 \\ 4xyz^2 & 2x^2z^2 - 2 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} 2y^2z^2 - 2 & 4xyz^2 & 4xy^2z \\ 4xyz^2 & 2x^2z^2 - 2 & 4x^2yz \\ 4xy^2z & 4x^2yz & 2x^2y^2 - 2 \end{pmatrix}$$

Consider $(0, 0, 0)$ Then:

$$H_1 = (-2)$$

$$H_2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

So $|H_1| = -2$, $|H_2| = 4$, $H_3 = -8$ so $(0, 0, 0)$ is a maximum.

Now consider $(\pm 1, \pm 1, \pm 1)$. Then:

$$H_1 = (0)$$

$$H_2 = \begin{pmatrix} 0 & \pm 4 \\ \pm 4 & 0 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} -0 & \pm 4 & \pm 4 \\ \pm 4 & 0 & \pm 4 \\ \pm 4 & \pm 4 & 0 \end{pmatrix}$$

as $|H_1| = 0$ we can't determine its nature from this test.