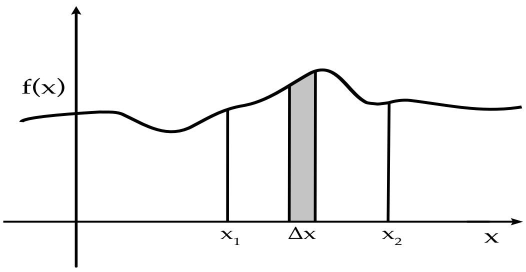
AS2051

Section 3: Integration

Integration of a function of two variables

Similar idea to one variable. Recall:



$$I = \int_{x_1}^{x_2} f(x)dx$$

For two variables:

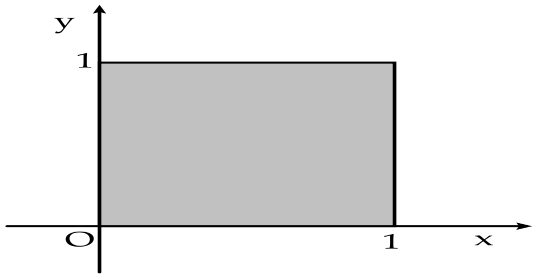
$$\begin{aligned} I &= \int_R f(x,y)dR \\ &= \iint\limits_R f(x,y)dxdy \end{aligned}$$

Example

Find the volume underneath the surface

$$f(x,y) = x^2 + 2xy + 1$$

where R is the square region:



Solution

$$\begin{aligned} I &= \int_0^1 \int_0^1 f(x,y)dxdy \\ &= \int_0^1 \left(\int_0^1 f(x,y)dx \right) dy \\ &= \int_0^1 \left(\int_0^1 (x^2 + 2xy + 1)\, dx \right) dy \\ &= \int_0^1 \left[\frac{x^3}{3} + x^2y + x \right]_0^1 dy \\ &= \int_0^1 \left(\frac{4}{3} + y \right) dy \\ &= \left[\frac{4}{3}y + \frac{y^2}{2} \right]_0^1 \\ &= \frac{11}{6} \end{aligned}$$

Note that the we could have performed the integration in the other order:

$$\begin{aligned} I &= \int_0^1 \int_0^1 f(x,y)dydx \\ &= \int_0^1 (x^2 + 2xy + 1)dydx \\ &= \int_0^1 [x^2y + xy^2 + y]_0^1 dx \\ &= \int_0^1 (x^2 + x + 1)dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 1 \right] \\ &= \frac{11}{6} \end{aligned}$$

More Complicated Regions

What about if the region R that we are interested in is not a square? Slice in the x direction, area of slice is:

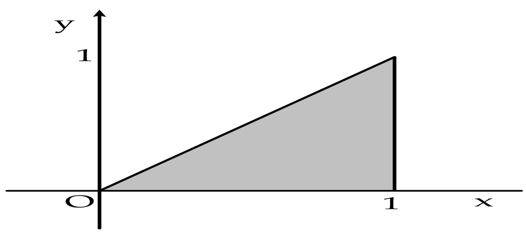
$$\int_{x_0}^{x_1} f(x,y)dx$$

where x_0 and x_1 will generally depend of y. So then we have:

$$\int_{y_0}^{y_1} \int_{x_0(y)}^{x_1(y)} f(x,y)dxdy$$

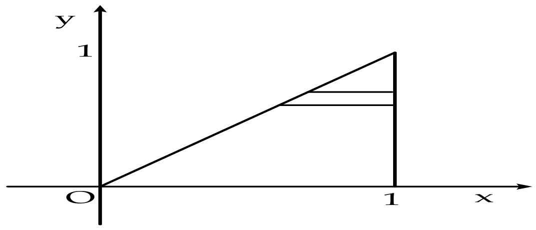
Example

Integrate $f = x^2 + 2xy + 1$ over the region



Solution

Initially choose to slice the region parallel to the x-axis:



So the inner integral is:

$$\int_y^1 f(x,y)dx$$

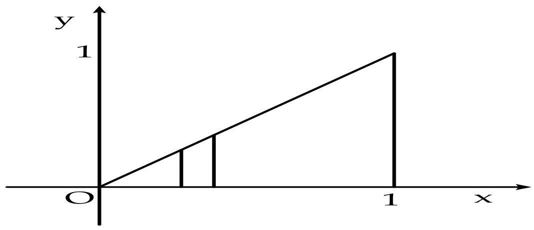
Now note that the slices are between $y = 0$ and $y = 1$. Therefore out integral is

$$I = \int_0^1 \left(\int_y^1 f(x,y)dx \right) dy$$

Now to perform the integration:

$$\begin{aligned}
I &= \int_0^1 \left(\int_y^1 f(x,y)dx \right) dy \\
&= \int_0^1 \left(\int_y^1 x^2 + 2xy + 1dx \right) dy \\
&= \int_0^1 \left[\frac{x^3}{3} + x^2y + x \right]_y^1 dy \\
&= \left(\frac{1}{3} + y + 1 - \frac{y^3}{3} - y^3 - y \right) dy \\
&= \left[\frac{y}{3} + \frac{y^2}{2} + y - \frac{y^4}{12} - \frac{y^4}{4} - \frac{y^2}{2} \right]_0^1 \\
&= \frac{1}{3} + \frac{1}{2} + 1 - \frac{1}{12} - \frac{1}{4} - \frac{1}{2} \\
&= 1
\end{aligned}$$

What about if we had chosen to slice parallel to the y-axis rather that the x-axis. In this case the inner integral is:



$$\int_0^x f(x,y)dy$$

and our slices are in the region between $x = 0$ and $x = 1$. Therefore our integral can be written as:

$$I = \int_0^1 \left(\int_0^x f(x,y)dy \right) dx$$

Now to perform the integration:

$$\begin{aligned}
I &= \int_0^1 \left(\int_0^x f(x,y)dy \right) dx \\
&= \int_0^1 \left(\int_0^x x^2 + 2xy + 1dy \right) dx \\
&= \int_0^1 \left[x^2y + xy^2 + y \right]_0^x dx \\
&= \int_0^1 2x^3 + xdx \\
&= \left[\frac{x^4}{2} + \frac{x^2}{2} \right]_0^1 \\
&= 1
\end{aligned}$$

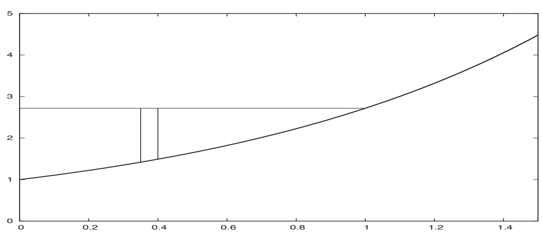
Example

Evaluate

$$I = \int_0^1 \int_{e^x}^e \frac{1}{\ln y} \cos\left(\frac{x}{\ln y} \right) dydx$$

Solution

Consider:



Can we make life easier by altering the order of integration?

So we can transform the integral given into.

$$\begin{aligned}
I &= \int_1^e \int_0^{\ln y} \frac{1}{\ln y} \cos\left(\frac{x}{\ln y} \right) dxdy \\
&= \int_1^e \left[\sin\left(\frac{x}{\ln y} \right) \right] dy \\
&= \int_1^e \sin(1)dy \\
&= [y \sin(1)]_1^e \\
&= (e-1)\sin(1)
\end{aligned}$$

Example

A cylinder in three dimensions is given by $x^2 + y^2 = 2ax$. If it is cut off by slices in the planes $z = 0$ and $z = mx$ find the volume. Note, it is actually possible to do this particular case without integration as follows: The centre of the top face is at $z = ma$, Cut the top part off and flip over to get a cylinder with volume $ma(\pi a^2)$ i.e. the volume is $m\pi a^3$.

Solution

First note that the height of each part above an area element $dxdy$ is mx. So the volume is going to be given by:

$$\begin{aligned} V &= \int_{x_0}^{x_1}\int_{y_0}^{y_1} mxdydx \\ &= \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} mxdydx \\ &= \int_0^{2a} [mxy]_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} dx \\ &= \int_0^{2a} 2mx\sqrt{2ax - x^2}dx \\ &= \int_0^{2a} 2mx\sqrt{a^2 - (x-a)^2}dx \end{aligned}$$

Now we change variables: $u = x - a$, remembering that we must also change the limits.

$$V = \int_{u=-a}^{a} 2m(u+a)\sqrt{a^2 - u^2}du$$

Now let $u = a\sin\theta$ and so $du = a\cos\theta d\theta$

$$\begin{aligned} V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2ma(\sin\theta + 1)a\cos\theta a\cos\theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2ma^3[\sin\theta\cos^2\theta + \cos^2\theta]d\theta \\ &= 2ma^3[I_1 + I_2] \end{aligned}$$

Now $I_1 = 0$.

$$\begin{aligned} I_2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2\theta + 1)d\theta \\ &= \left[\frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \end{aligned}$$

so

$$V = 2ma^3(\frac{\pi}{2}) = \pi ma^3$$

This was rather time consuming. Is the a better way to deal with the boundary?

Consider changing coordinates from Cartesian to polar. Let

$$x = a + r\cos\theta$$

$$y = r\sin\theta$$

then the height is $m(a + r\cos\theta)$

$$V = \int_0^{2\pi}\int_0^a m(a + r\cos\theta)rdrd\theta$$

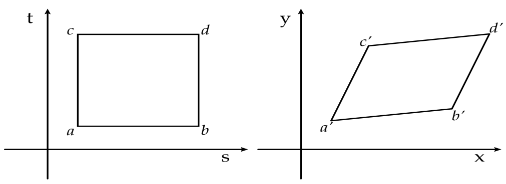
Using a small area element $rdrd\theta$

$$\begin{aligned} V &= \int_0^{2\pi}\left[\frac{mar^2}{2} + \frac{mr^3}{3}\cos\theta\right]_0^a d\theta \\ &= \int_0^{2\pi}\left[\frac{ma^3}{2} + \frac{ma^3}{3}\cos\theta\right]_0^a d\theta \\ &= \pi ma^3 \end{aligned}$$

Changing Variables In Integration

In the last example we saw that things became a lot easier in the integration when we changed at the start from Cartesian coordinates to polar. The area element in these co-ordinates was $dxdy$ and $rdrd\theta$ respectively.
This raises the question of how do we make such changes in general both in two dimensions and three dimensions?

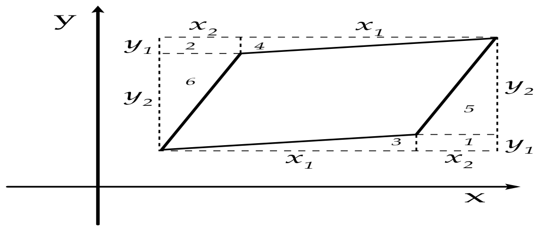
Consider $x(s,t)$ and $y(s,t)$.



The lower left coordinate a is, say, at (s_0, t_0). This will be mapped to a' which is at $x_0 = x(s_0, t_0), y_0 = y(s_0, t_0)$.

Consider now the mapping of the other coordinates b, c, d.
The bottom right coordinate b is mapped to $(x(s+\delta s), y(s+\delta s))$. b' is at $(x_0 + \delta s\frac{\partial x}{\partial s}, y_0 + \delta s\frac{\partial y}{\partial s})$.
Similarly c' is at $(x_0 + \delta t\frac{\partial x}{\partial t}, y_0 + \delta s\frac{\partial y}{\partial t})$.
d' is $(x_0 + \delta s\frac{\partial x}{\partial s} + \delta t\frac{\partial x}{\partial t}, y_0 + \delta s\frac{\partial y}{\partial s} + \delta t\frac{\partial y}{\partial t})$.

Now our area in $x - y$ space is:



$$\begin{aligned} A &= (x_1 + x_2)(y_1 + y_2) - x_2 y_1 - x_2 y_1 - \frac{1}{2} x_1 y_1 \\ &- \frac{1}{2} x_1 y_1 - \frac{1}{2} x_2 y_2 - \frac{1}{2} x_2 y_2 \\ &= x_1 y_2 - x_2 y_1 \end{aligned}$$

Now

$$x_1 = \delta s \frac{\partial x}{\partial s} \quad x_2 = \delta t \frac{\partial x}{\partial t} \quad y_1 = \delta s \frac{\partial y}{\partial s} \quad y_2 = \delta t \frac{\partial y}{\partial t}$$

$$\begin{aligned} A &= x_1 y_2 - x_2 y_1 \\ &= \delta s \frac{\partial x}{\partial s} \delta t \frac{\partial y}{\partial t} - \delta t \frac{\partial x}{\partial t} \delta s \frac{\partial y}{\partial s} \\ &= \delta s \delta t \left(\frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right) \end{aligned}$$

In general
$dxdy = |J| dsdt$
where the Jacobian of the transformation is:

$$J = \left| \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \right| = \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s}$$

Polar Coordinates

Recall $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\begin{aligned} J &= \left| \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| \end{aligned}$$

Therefore $|J| = rcos^2\theta + r\sin^2\theta$ i.e. $|J| = r$. So $dxdy = rdrd\theta$

Three or More Variables

Now if, for example, $x(r, s, t)$, $y(r, s, t)$ and $z(r, s, t)$.
Then

$$dxdydz = |J| drdsdt$$

where

$$J = \left| \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} \right|$$

This naturally extends to higher number of variables.

Example

Find the volume underneath $f(x, y) = \sqrt{1 - (x^2 + y^2)}$ in the region bounded by the unit disc.

Solution

$$I = \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} rdrd\theta$$

Let $r = \sin\phi$

$$I = \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \cos\phi d\phi d\theta$$

However $\cos\phi \sin\phi \cos\phi$ can be written as $sin\phi - \sin^3\phi$, which can in turn be written as $\frac{1}{4}(\sin 3\phi + \sin\phi)$. Therefore:

$$\begin{aligned} I &= \int_0^{2\pi} \left[\frac{1}{4} \left(-\frac{\cos 3\phi}{3} - \cos\phi \right) \right]_0^{\frac{\pi}{2}} d\theta \\ &= \frac{2\pi}{3} \end{aligned}$$

Example

Calculate

$$I = \int_0^\infty e^{-x^2} dx$$

Solution

$$\begin{aligned} I^2 &= II \\ &= \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy \\ &= \int_0^\infty \int_0^\infty e^{-x^2-y^2}dxdy \\ &= \iint_R e^{-(x^2+y^2)}dxdy \end{aligned}$$

$$\begin{aligned} I^2 &= \iint_R e^{-(r^2)}rdrd\theta \\ &= \int_0^{\pi/2}\int_0^\infty e^{-(r^2)}rdrd\theta \\ &= \int_0^{\pi/2}[-\frac{1}{2}e^{-(r^2)}]_0^\infty d\theta \\ &= \frac{\pi}{4} \end{aligned}$$

so $I = \frac{\sqrt{\pi}}{2}$

Other Coordinate Systems

There are other coordinate systems. The two most common in three dimensions are:

- Cylindrical Polars
- Spherical Polars

Cylindrical Polar Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

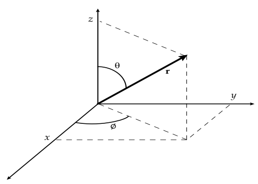
$$z = z$$

$$\begin{aligned} J &= \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| \end{aligned}$$

So $|J| = r$. and so for a function, f

$$I = \iiint_R frdrd\theta dz$$

Spherical Polar Coordinates



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\begin{aligned} J &= \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}\right| \end{aligned}$$

$$\begin{aligned} J &= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) + r\cos\theta\cos\phi(r\sin\theta\cos\theta\cos\phi) \\ &+ r\sin\theta\sin\phi(r\sin^2\theta\sin\phi + r\cos^2\theta\sin\phi) \\ &= r^2(\sin^3\theta\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin^3\theta\sin^2\phi \\ &\quad + \sin\theta\cos^2\theta\sin^2\phi) \\ &= r^2(\sin^3\theta + \sin\theta\cos^2\theta) \\ &= r^2\sin\theta(\sin^2\theta + \cos^2\theta) \\ &= r^2\sin\theta \end{aligned}$$

$$I = \iiint_R fr^2\sin\theta drd\phi d\theta$$

Example

Find the volume of a sphere of radius a.

Solution

$$
\begin{aligned}
I &= \iiint_V dxdydz \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^{a} r^2 \sin\theta dr d\phi d\theta \\
&= \int_0^{\pi} \int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^{a} d\phi d\theta \\
&= \int_0^{\pi} \int_0^{2\pi} \frac{a^3}{3}\sin\theta d\phi d\theta \\
&= \int_0^{\pi} \left[\phi\frac{a^3}{3}\sin\theta\right]_0^{2\pi} d\theta \\
&= \int_0^{\pi} 2\pi\frac{a^3}{3}\sin\theta d\theta \\
&= \frac{4\pi a^3}{3}
\end{aligned}
$$