AS2051

Section 4: Laplace Transforms

Definition

The Laplace transform of a function $f(x)$ is defined for $x \geq 0$ as:

$$F\{p\} = \mathfrak{L}(f) = \int_0^\infty f(x)e^{-px}dx$$

Note that for a constant, k, and a function, f:

$$\mathfrak{L}(kf) = k\mathfrak{L}(f)$$

and for two functions f and g:

$$\mathfrak{L}(f+g) = \mathfrak{L}(f) + \mathfrak{L}(g)$$

Example 1

$$\begin{aligned}\mathfrak{L}(1) &= \int_0^\infty e^{-px}dx \\ &= \left[-\frac{1}{p}e^{-px}\right]_0^\infty\end{aligned}$$

for $(p > 0)$

$$\mathfrak{L}(1) = \frac{1}{p}$$

Example 2

$$\begin{aligned}\mathfrak{L}(e^{ax}) &= \int_0^\infty e^{ax}e^{-px}dx \\ \mathfrak{L}(e^{ax}) &= \int_0^\infty e^{-(p-a)x} \\ &= \left[-\frac{1}{p-a}e^{-(p-a)x}\right]_0^\infty\end{aligned}$$

for $(p > a)$

$$\mathfrak{L}(e^{ax}) = \frac{1}{p-a}$$

Example 3

$$\begin{aligned}\mathfrak{L}(x^n) &= \int_0^\infty x^n e^{-px}dx \\ &= \left[-\frac{1}{p}e^{-px}x^n\right]_0^\infty + \frac{1}{p}\int_0^\infty nx^{n-1}e^{-px}dx \\ &= \left[-\frac{1}{p}e^{-px}x^n\right]_0^\infty + \frac{n}{p}\mathfrak{L}(x^{n-1}) \\ &= \frac{n}{p}\mathfrak{L}(x^{n-1}) \\ &= \frac{n}{p}\frac{n-1}{p}\mathfrak{L}(x^{n-2}) \\ &= \frac{n!}{p^n}\mathfrak{L}(x^0) \\ &= \frac{n!}{p^{n+1}}\end{aligned}$$

Example 4

$$\begin{aligned}\mathfrak{L}(\cos(ax)) &= \int_0^\infty \cos(ax)e^{-px}dx \\ &= \left[\frac{1}{a}\sin(ax)e^{-px}\right]_0^\infty - \frac{1}{a}\int_0^\infty \sin(ax)e^{-px}(-p)dx \\ &= \frac{p}{a}\mathfrak{L}(\sin(ax))\end{aligned}$$

But what is $\mathfrak{L}(\sin(ax))$?

Example 5

$$\begin{aligned}\mathfrak{L}(\sin(ax)) &= \int_0^\infty \sin(ax)e^{-px}dx \\ &= \left[-\frac{1}{a}\cos(ax)e^{-px}\right]_0^\infty + \frac{1}{a}\int_0^\infty \cos(ax)e^{-px}(-p)dx \\ &= \frac{1}{a} - \frac{p}{a}\mathfrak{L}(\cos(ax)) \\ &= \frac{1}{a} - \frac{p^2}{a^2}\mathfrak{L}(\sin(ax))\end{aligned}$$

Rearranging we get:

$$\mathfrak{L}(\sin(ax)) = \frac{a}{p^2+a^2}$$

Substituting this into what we had for example 4 we can then write:

$$\mathfrak{L}(\cos(ax)) = \frac{p}{p^2+a^2}$$

Solving Differential Equations

If we want to solve differential equations we need to know the Laplace transform of f', f'' etc.

$$\begin{aligned}\mathfrak{L}(f'(x)) &= \int_0^\infty f'(x)e^{-px}dx \\ &= \left[f(x)e^{-px}\right]_0^\infty - \int_0^\infty f(x)(-p)e^{-px}dx \\ &= -f(0) + p\int_0^\infty f(x)e^{-px}dx \\ &= -f(0) + p\mathfrak{L}(f(x))\end{aligned}$$

$$\begin{aligned}\mathfrak{L}(f''(x)) &= -f'(0) + p\mathfrak{L}f'(x) \\ &= -f'(0) - pf(0) + p^2\mathfrak{L}(f(x))\end{aligned}$$

Example

Solve

$$f'' + 3f' + 2f = 1$$

subject to f(0)=0, f'(0)=0.

Solution

Take the Laplace transform of the equation and use the notation $F\{p\} = \mathfrak{L}(f(x))$:

$$-f'(0) - pf(0) + p^2 F\{p\} + 3(-f(0) + pF\{p\}) + 2F\{p\} = \frac{1}{p}$$

$$(p^2 + 3p + 2)F\{p\} = \frac{1}{p}$$

$$F\{p\} = \frac{1}{p(p+1)(p+2)}$$

This can be written as:

$$F\{p\} = \frac{1}{2p} - \frac{1}{p+1} + \frac{1}{2(p+2)}$$

Now carry out the inverse transform:

$$f(x) = \frac{1}{2} - e^{-x} + \frac{1}{2}e^{-2x}$$

Shifting Theorems

So far we have been discussing $f(x)$. However, often Laplace transforms are used in problems where $f(t)$ where:

$$\mathfrak{L}(f(t)) = \int_0^\infty f(t)e^{-pt}dt$$

These problems are often concerned with the behaviour after an initial point in time. What happens to the Laplace Transform if the function starts at $t = a$ rather than $t = 0$?

t-shifting theorem

Consider $y(t) = 0$ for $0 \le t < a$ and $y(t) = f(t-a)$ for $a \le t$.

$$\begin{aligned}\mathfrak{L}(y(t)) &= \int_0^\infty y(t)e^{-pt}dt \\ &= \int_0^a 0e^{-pt}dt + \int_a^\infty f(t-a)e^{-pt}dt\end{aligned}$$

Let $t' = t - a$:

$$\begin{aligned}\mathfrak{L}(y(t)) &= \int_0^\infty f(t')e^{-p(t'+a)}dt \\ &= e^{-pa}\int_0^\infty f(t')e^{-pt'}dt' \\ &= e^{-pa}\mathfrak{L}(f(t))\end{aligned}$$

Heaviside Step Function

Note that we can write $y(t)$ as

$$y(t) = H(t-a)f(t-a)$$

where H is the Heaviside step function, which is defined as follows: $H(t) = 1$ for $t \ge 0$ and $H(t) = 0$ for $t < 0$.
So

$$\mathfrak{L}(H(t-a)f(t-a)) = e^{-pa}\mathfrak{L}(f(t))$$

p-shifting theorem

$$\begin{aligned}\mathfrak{L}(e^{ax}f(x)) &= \int_0^\infty e^{ax}f(x)e^{-px}dx \\ &= \int_0^\infty f(x)e^{-(p-a)x}dx \\ &= F\{p-a\}\end{aligned}$$

Examples

We know

$$\mathfrak{L}(1) = \frac{1}{p}$$

and so, using the p-shifting theorem we have

$$\mathfrak{L}(e^{ax}) = \frac{1}{p-a}$$

(which we already knew). However, now consider that we know:

$$\mathfrak{L}(x) = \frac{1}{p^2}$$

Then we can now easily, using the p-shifting theorem, obtain:

$$\mathfrak{L}(xe^{ax}) = \frac{1}{(p-a)^2}$$

Similarly we know

$$\mathfrak{L}(x^n) = \frac{n!}{p^{n+1}}$$

and so, using the p-shifting theorem, we have:

$$\mathfrak{L}(x^n e^{ax}) = \frac{n!}{(p-a)^{n+1}}$$

Example

Solve

$$\frac{d^2 f(t)}{dt^2} + 3\frac{df(t)}{dt} + 2f(t) = 2r(t)$$

where

$$\frac{df(0)}{dt} = f(0) = 0$$

and

$$r(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}$$

Solution

We first need to determine the Laplace transform of the righthand side of the equation. This can be obtained in two ways.
1) Directly

$$\begin{aligned} \mathfrak{L}(r(t)) &= \int_0^\infty r(t)e^{-pt}dt \\ &= \int_0^1 r(t)e^{-pt}dt \\ &= \left[-\frac{1}{p}e^{-pt}\right]_0^1 \\ &= \frac{1}{p}(1-e^{-p}) \end{aligned}$$

2) Write $r(t) = 1 - H(t-1)$.
Then, using the t-shifting theorem:

$$\mathfrak{L}(r) = \frac{1}{p} - \frac{1}{p}e^{-p}$$

The Laplace transform of the whole equation can now be determined:

$$p^2F\{p\} + 3pF\{p\} + 2F\{p\} = \frac{2}{p}(1-e^{-p})$$

Note that this has been obtained using the conditions given i.e. $f'(0) = f(0) = 0$. This can we written as:

$$F(p) = \frac{2}{p(p+1)(p+2)}(1-e^{-p})$$

We now need to perform the inverse transform. We first note that

$$\frac{2}{p(p+1)(p+2)} = \frac{1}{p} - \frac{2}{p+1} + \frac{1}{p+2}$$

Now the inverse Laplace transform of this is

$$1 - 2e^{-t} + e^{-2t}$$

Therefore

$$f(t) = 1 - 2e^{-t} + e^{-2t} - H(t-1)[1 - 2e^{-(t-1)} + e^{-2(t-1)}]$$

Example

Solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = \cos x$$

subject to $y(0) = y'(0) = 0$

Solution

Let Y denote the Laplace Transform of y i.e. $\mathfrak{L}(y)$. Then we can transform the equation given to obtain

$$p^2Y + 4pY + 8Y = \frac{p}{p^2+1}$$

and so

$$Y = \frac{p}{(p^2+1)(p^2+4p+8)}$$

It is tempting to try to express this as:

$$Y = \frac{ap+b}{p^2+1} + \frac{cp+d}{p^2+4p+8}$$

However, in this context, it is better to write this in the form:

$$Y = \frac{Ap+B}{p^2+1} + \frac{C(p+2)+D}{(p+2)^2+4}$$

We need to determine A, B, C, D. We have:

$$p = (Ap+B)(p^2+4p+8) + (C(p+2)+D)(p^2+1)$$

or

$$p = (A+C)p^3 + (4A+B+2C+D)p^2 + (8A+4B+C)p + (8B+2C+D)$$

So we have the following set of equations to solve

$$0 = A + C$$

$$0 = 4A + B + 2C + D$$

$$1 = 8A + 4B + C$$

$$0 = 8B + 2C + D$$

These can be solved to give:

$$D = -\frac{18}{65}$$

$$A = \frac{7}{65}$$

$$C = -\frac{7}{65}$$

$$B = \frac{4}{65}$$

We therefore have

$$Y\{p\} = \frac{7}{65}\frac{p}{p^2+1} + \frac{4}{65}\frac{1}{p^2+1} - \frac{7}{65}\frac{p+2}{(p+2)^2+4} - \frac{9}{65}\frac{2}{(p+2)^2+4}$$

Inverse transform to get:

$$y = \frac{7}{65}\cos x + \frac{4}{65}\sin x - \frac{7}{65}e^{-2x}\cos(2x) - \frac{9}{65}e^{-2x}\sin(2x)$$

Note

Consider $Y\{p\} = F\{p\}G\{p\}$. If we know $f(x) = \mathfrak{L}^{-1}F$ and $g(x) = \mathfrak{L}^{-1}G$ can we find $y(x)$? Consider the convolution integral

$$q(x) = \int_0^x f(x-x')g(x')dx'$$

The Laplace transform of q is:

$$Q = \int_0^\infty \int_0^x f(x-x')g(x')dx' e^{-px}dx$$

Change the order of integration:

$$\begin{aligned} Q &= \int_0^\infty \int_{x'}^\infty f(x-x')g(x')e^{-px}dxdx' \\ &= \int_0^\infty g(x') \int_{x'}^\infty f(x-x')e^{-px}dxdx' \end{aligned}$$

Let $x'' = x - x'$
Then

$$\begin{aligned} Q\{p\} &= \int_0^\infty g(x') \int_0^\infty f(x'')e^{-px''}e^{-px'}dx''dx' \\ &= \int_0^\infty g(x')e^{-px'} \int_0^\infty f(x'')e^{-px''}dx''dx' \\ &= \int_0^\infty g(x')e^{-px'}F\{p\}dx' \\ &= G\{p\}F\{p\} \end{aligned}$$

So if the Laplace transform is the product of two known Laplace transforms we can express its inverse as a convolution.

Simple Example

Consider

$$\frac{1}{p^4} = \frac{1}{p^2}\frac{1}{p^2}$$

Determine the inverse Laplace transform of

$$\frac{1}{p^4}$$

using the convolution integral.

We know $f(x) = \mathfrak{L}^{-1}F = x$ and $g(x) = \mathfrak{L}^{-1}G = x$.
The convolution integral is:

$$\begin{aligned} \int_0^x (x-x')x'dx' &= \left[\frac{xx'^2}{2} - \frac{x'^3}{3}\right]_0^x \\ &= \frac{x^3}{2} - \frac{x^3}{3} \\ &= \frac{x^3}{6} \end{aligned}$$

so

$$\mathfrak{L}^{-1}\left(\frac{1}{p^4}\right) = \frac{x^3}{6}$$

Note that this example can be checked with the formula we had earlier in the notes.

$$\begin{aligned} \mathfrak{L}(x^3) &= \frac{n!}{p^{n+1}} \\ &= \frac{3!}{p^4} \\ &= \frac{6}{p^4} \end{aligned}$$