AS2051

Section 5: More Differential Equations

Order of a Differential Equation

The order of a differential equation is the highest order derivative that appears. An example of a first order equation is:

$$\frac{dy}{dx} = f(x)$$

An example of a second order equation is

$$\ddot{x} + \dot{x} = 0$$

Linear and Nonlinear Equations

An n^{th} order differential equation for $y(x)$ is linear if it can be written in the form:

$$a_n(x)\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + \ldots\ldots + a_1(x)\frac{dy(x)}{dx} + a_0(x)y(x) = f(x)$$

If it can not be written in this form then it is said to be nonlinear.
If $f(x) = 0$ then the equation is said to be homogeneous and it is said to be inhomogeneous if $f(x) \neq 0$.

Why is linearity important?

If we have a linear homogeneous differential equation e.g.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

and two solutions y_1 and y_2, then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution. This can be checked as follows:

$$\begin{aligned}
\frac{d^2(\alpha_1 y_1 + \alpha_2 y_2)}{dx^2} + x\frac{d(\alpha_1 y_1 + \alpha_2 y_2)}{dx} - \alpha_1 y_1 + \alpha_2 y_2 &= 0 \\
\alpha_1\left(\frac{d^2 y_1}{dx^2} + x\frac{dy_1}{dx} - y_1\right) + \alpha_2\left(\frac{d^2 y_2}{dx^2} + x\frac{dy_2}{dx} - y_2\right) &= 0 \\
\alpha_1 0 + \alpha_2 0 &= 0 \\
0 &= 0
\end{aligned}$$

Linear Second Order ODEs

Consider the equation

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = d(x)$$

When $d(x) = 0$ then we have the homogeneous equation:

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

The inhomogeneous linear ODE has the property:If y_1 and y_2 are two independent solutions of the homogeneous equation and y_p is a solution of the inhomogeneous equation then:

$$y_{tot} = y_p + Ay_1 + By_2$$

is a solution of the inhomogeneous equation for constants A and B.

Example

Solve

$$y'' - y = 1$$

Solution

First we consider

$$y'' - y = 0$$

and see solutions of the form

$$y(x) \sim e^{\lambda x}$$

for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.
Differentiate y and substitute into the homogeneous equation gives:

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0$$

Now $e^{\lambda x} \neq 0$ so

$$\lambda^2 - 1 = 0$$

We refer to this as the auxiliary equation.

We solve the auxiliary equation to get $\lambda_1 = 1$, $\lambda_2 = -1$. Therefore

$$y_{cf} = Ae^{x} + Be^{-x}$$

Now we need to determine the particular solution.
We try $y = c$
Substituting into the equation we determine that $c = -1$.
Therefore the full solution is:

$$y = Ae^{x} + Be^{-x} - 1$$

Operator Notation

If we have a function $f(x)$ we can write $F(f(x))$ for a function F of $f(x)$ e.g. $F(f(x)) = f(x) + 2f^2(x)$. We can use similar notation for an operator, which performs other manipulations of f.
Recall that we have already met this notation once before with Laplace transforms:

$$\mathfrak{L}(f) = \int_0^\infty f(x)e^{-px}dx$$

For linear differential equations we could write

$$\frac{d}{dx} \equiv D$$

Then we can write a differential equation concisely as:

$$P(D)f = q(x)$$

where P is a polynomial.

Example

Solve

$$P(D)f = e^{3x}$$

where $P(z) = z^2 + 3z + 2$.

Solution

First we note that our equation can be written more fully as:

$$\begin{aligned} P(D)f &= 0 \\ (D^2 + 3D + 2)f &= 0 \\ \frac{d^2f}{dx^2} + 3\frac{df}{dx} + 2f &= 0 \end{aligned}$$

Recall that for a homogeneous equation we seek solutions of the form $f(x) = e^{\lambda x}$. We then obtain, written in a concise form

$$P(\lambda)e^{\lambda x} = 0$$

From which we deduce that $P(\lambda) = 0$ i.e. $\lambda^2 + 3\lambda + 2 = 0$ and so

$$y = Ae^{-x} + Be^{-2x}$$

Constant Coefficient Ordinary Differential Equations (In General)

Consider

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_2\frac{dy^2}{dx^2} + a_1\frac{dy}{dx} + a_0y = f(x)$$

where $a_n, a_{n-1}, a_{n-2}, \ldots\ldots, a_2, a_1, a_0$ are constants. This is an n^{th} order differential equation. This can be concisely written and

$$P(D)y = f(x)$$

where

$$P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots\ldots a_2z^2 + a_1z + a_0$$

To find solutions of the homogeneous equation, consider the roots of $P(\lambda) = 0$

In general there will be n roots. However, these roots can be

- complex
- repeated

If $P(z) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)\ldots\ldots(z - \lambda_n)$
where all the λ_i are distinct, then the general solution to

$$P(D)f = 0$$

is

$$f = \sum_{i=1}^{n} A_ie^{\lambda_ix}$$

Second Order Linear Ordinary Differential Equations with Non-Constant coefficients

Consider

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = R(x)$$

We begin trying to solve this equation by first finding solutions to the homogeneous equation:

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$$

Existence Theorem

Let

$$L(y) = \frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y$$

where p_1 and p_2 are continuous function in an interval I. Let x_0 to be a point in the interval I. Then there exists a $y = f(x)$ such that

$$L(y) = 0$$

and
$y(x_0) = a$, $y'(x_0) = b$ where a, b are real numbers.

Uniqueness Theorem

If $f(x)$ and $g(x)$ are two solutions of the given ordinary differential equation i.e. $L(f) = 0$ and $L(g) = 0$ and both satisfy

$$f(x_0) = g(x_0)$$

$$f'(x_0) = g'(x_0)$$

for some x_0 in I. Then $f(x) = g(x)\ \forall x \in I$

Characterization of Solutions

Consider

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$$

with p_1, p_2 continuous on an interval $I = \mathbb{R}$. Let $y_1(x)$ and $y_2(x)$ be two non-zero functions satisfying $L(y_1) = 0$ and $L(y_2) = 0$ in I such that $y_1(x)/y_2(x)$ is not a constant.

Then

- $$y = c_1y_1(x) + c_2y_2(x)$$
  (where c_1, c_2 are constants) is a solution to $L(y) = 0$
- Conversely if we can find a solution y to $L(y) = 0$ then we can find c_1 and c_2 such that
  $$y = c_1y_1(x) + c_2y_2(x)$$
  i.e. all the solutions can be expressed in terms of y_1 and y_2.

The first of these two points is just a statement of linearity. For the second point:
Consider a solution $y(x)$ to $L(y) = 0$ and choose a point $x_0 \in I$.
Now let $y(x_0) = a$ and $y'(x_0) = b$. Try to find c_1 and c_2 such that

$$a = c_1y_1(x_0) + c_2y_2(x_0)$$

and

$$b = c_1y_1'(x_0) + c_2y_2'(x_0)$$

We can write these two equations as:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Solutions will exist if

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

So if there is a point where this determinant is non-zero then we can find a solution.

Can there be no such point?

Then

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

$\forall x \in I$ i.e. $y_1y_2' - y_2y_1' = 0$.
Therefore

$$\frac{y_2'}{y_2} = \frac{y_1'}{y_1}$$

Integrating we get

$$\ln y_2 = \ln y_1 + \ln c$$

or y_2/y_1 is a constant which breaks our assumption stated earlier.

As this contradicts our assumption there must be a point where

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

Note that we call

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$

the Wronskian i.e.

$$W(x) = y_1y_2' - y_2y_1'$$

Consider now the inhomogeneous equation

$$L(y) = y'' + p_1(x)y' + p_2(x)y = r(x)$$

and two solutions $y = f(x)$ and $y = g(x)$. Then

$$L(g - f) = L(g) - L(f) = r(x) - r(x) = 0$$

and so

$$g - f = c_1y_1(x) + c_2y_2(x)$$

or

$$g = f + c_1y_1(x) + c_2y_2(x)$$

So if we have a particular solution $f(x)$ to the inhomogeneous equation and y_1, y_2 are linearly independent solutions then a general solution will be

$$y = f + c_1y_1(x) + c_2y_2(x)$$

The Methods of Variation of Parameters

We want to find the solution to:

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x)$$

Let y_1 and y_2 be solutions of $L(y) = 0$ and let

$$W(x) = y_1 y_2' - y_2 y_1'$$

Then the particular solution of the inhomogeneous equation is of the from

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = -\int \frac{y_2 R(x)}{W(x)} dx$$

$$u_2(x) = \int \frac{y_1 R(x)}{W(x)} dx$$

Example

Find the solution of

$$y'' - y = x$$

Solution

Using the techniques that you learned last year we can determine that the solution to the example is

$$y = Ae^x + Be^{-x} - x$$

However, does our new technique work to find the particular integral for this simple example?

Let $y_1 = e^x$ and $y_2 = e^{-x}$

Now

$$\begin{aligned} W(x) &= y_1 y_2' - y_2 y_1' \\ &= -e^x e^{-x} - e^x e^{-x} \\ &= -2 \end{aligned}$$

$$\begin{aligned} u_1 &= \int \frac{e^{-x}x}{-2} dx \\ &= -\frac{x}{2}e^{-x} + \int \frac{e^{-x}}{2} dx \\ &= -\frac{x}{2}e^{-x} - \frac{e^{-x}}{2} + c \end{aligned}$$

$$\begin{aligned} u_2 &= \int \frac{e^x x}{-2} dx \\ &= -\frac{x}{2}e^x + \int \frac{e^x}{2} dx \\ &= -\frac{x}{2}e^x + \frac{e^x}{2} + d \end{aligned}$$

and so

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(-\frac{x}{2}e^{-x} - \frac{e^{-x}}{2} + c\right)e^x + \left(-\frac{x}{2}e^x + \frac{e^x}{2} + d\right)e^{-x} \\ &= \frac{-x}{2} - \frac{1}{2} - \frac{x}{2} + \frac{1}{2} + ce^x + de^{-x} \\ &= -x + ce^x + de^{-x} \end{aligned}$$

So out solution is

$$y = Ae^x + Be^{-x} - x + ce^x + de^{-x}$$

or $y = A'e^x + B'e^{-x} - x$

Example

Find the solution of

$$y'' + y = \sec x$$

Solution

First consider the homogeneous equation for which we obtain
$y_1 = \cos x$ $y_2 = \sin x$
and we obtain

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

We now proceed to form $y = u_1 y_1 + u_2 y_2$ by first obtaining u_1 and u_2

$$\begin{aligned} u_1 &= -\int \frac{\sec x \sin x}{1} dx \\ &= -\int \frac{\sin x}{\cos x} dx \\ &= \ln|\cos x| + c \end{aligned}$$

$$\begin{aligned} u_2 &= \int \frac{\sec x \cos x}{1} dx \\ &= \int 1 dx \\ &= x + d \end{aligned}$$

$$\begin{aligned} y &= y_1(x)u_1(x) + y_2(x)u_2(x) \\ &= \cos x(\ln|\cos x| + c) + \sin x(x + d) \end{aligned}$$

Then

$$y_{tot} = A\cos x + B\sin x + \cos x \ln|\cos x| + x\sin x$$

Deriving the expressions for u_1 and u_2

Seek solutions to the equation:

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x)$$

of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 are solutions to the homogeneous equation.
We additionally specify that

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0$$

.

Now differentiate $y = u_1(x)y_1(x) + u_2(x)y_2(x)$ with respect to x.

$$\begin{aligned}\frac{dy}{dx} &= u_1'y_1 + u_1(x)y_1'(x) + u_2'y_2 + u_2y_2' \\ &= u_1y_1' + u_2y_2'\end{aligned}$$

similarly

$$\frac{d^2y}{dx^2} = u_1'y' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Now substitute these expressions into our equation:

$$\begin{aligned}\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2y &= R(x) \\ u_1'y' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p_1(u_1y_1' + u_2y_2') & \\ +p_2(u_1(x)y_1(x) + u_2(x)y_2(x)) &= \\ u_1'y_1' + u_2'y_2' + u_1(y_1'' + p_1y_1' + p_2y_1) + u_2(y_2'' + p_1y_2' + p_2y_2) &= \\ u_1'y_1' + u_2'y_2' &= R(x)\end{aligned}$$

So then we have the following pair of equations to solve.

$$u_1'y_1' + u_2'y_2' = R(x)$$

$$u_1'y_1 + u_2'y_2 = 0$$

Solving to find u_1':

$$u_1'y_1'y_2 - u_1'y_2'y_1 = y_2R(x)$$

$$-u_1'W(x) = y_2R(x)$$

and so

$$u_1(x) = -\int \frac{y_2R(x)}{W(x)}dx$$

By a similar method you obtain

$$u_2(x) = \int \frac{y_1R(x)}{W(x)}dx$$

Example

Find the general solution for

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = x\sec^2 x$$

Solution

First consider

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$$

Seek solutions of the form $y = x^n$

$$\begin{aligned}n(n-1)x^{n-2} - 2n\frac{x^{n-1}}{x} - 2\frac{x^n}{x^2} &= 0 \\ x^{n-2}(n(n-1) + 2n + 2) &= 0\end{aligned}$$

$x^{n-2} \neq 0$ and so $(n^2 - 3n + 2) = 0$ or $(n-1)(n-2) = 0$.
Let $y_1 = x$ and $y_2 = x^2$.

Therefore

$$\begin{aligned}W(x) &= x(2x) - 1x^2 \\ &= x^2\end{aligned}$$

Now we need to evaluate u_1 and u_2.

$$\begin{aligned}u_1(x) &= -\int \frac{y_2R(x)}{W(x)}dx \\ &= -\int \frac{x^2(x\sec^2 x)}{x^2}dx \\ &= -\int x\sec^2 x dx \\ &= -x\tan x + \int \tan x dx \\ &= -x\tan x - \ln|\cos x| + c\end{aligned}$$

$$\begin{aligned} u_2(x) &= \int \frac{y_1 R(x)}{W(x)} dx \\ &= -\int \frac{x(x\sec^2 x)}{x^2} dx \\ &= -\int \sec^2 x dx \\ &= \tan x + d \end{aligned}$$

Therefore

$$\begin{aligned} y &= (-x\tan x - \ln|\cos x| + c)x + (\tan x + d)x^2 \\ &= -x\ln|cosx| + cx + dx^2 \end{aligned}$$

So our general solution can be written as:

$$y = Ax + Bx^2 - x\ln|\cos x|$$

Note: You can check that $y = Ax + Bx^2 - x\ln(\cos x)$ is really a solution and it is good practice to do so. To check this differentiate:

$$y' = -\ln|\cos x| + x\tan x + A + (2B)x$$

$$y'' = \tan x + \tan x + x\sec^2 x + 2B$$

then substitute into the equations you were given and show that LHS=RHS:

$$\begin{aligned} LHS &= y'' - \frac{2y'}{x} + \frac{2y}{x^2} \\ &= 2\tan x + x\sec^2 x + 2B \\ &\quad -\frac{2}{x}(-\ln|\cos x| + x\tan x + A + (2B)x) \\ &\quad +\frac{2}{x^2}(-x\ln|\cos x| + Ax + Bx^2) \\ &= x\sec^2 x \\ &= RHS \end{aligned}$$

How do we find both y_1 and y_2?

The method of variation of parameters requires us to know both of y_1 and y_2. There is a technique that can help you to achieve this goal if you know one solution to the homogeneous equation but do not know the other.

Finding a Second Solution to a Homogeneous Equation

Assume that we know one solution to

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x)$$

we try to seek a solution of the form $u(x)y_1(x)$ where we do not yet know $u(x)$.
We substitute this expression for y into the homogeneous equation.

$$\begin{aligned} u''y_1 + 2u'y_1' + uy_1'' + p_1u'y_1 + p_1uy_1' + p_2uy_1 &= 0 \\ u''y_1' + 2u'y_1' + p_1u'y_1 + u(y_1'' + p_1y_1' + p_2y_1) &= 0 \end{aligned}$$

and since y_1 is a solution to the differential equation the term in brackets is zero giving us

$$u''y_1' + 2u'y_1' + p_1u'y_1 = 0$$

Let $v = u'$ then

$$v'y_1 + 2vy_1' + p_1vy_1 = 0$$

This can be written as:

$$\frac{v'}{v} = -2\frac{y_1'}{y_1} - p_1$$

Integrating we obtain:

$$\ln v = -2\ln y_1 - \int p_1 dx$$

or

$$v = \frac{e^{-\int p_1 dx}}{y_1^2}$$

Therefore

$$u' = \frac{e^{-\int p_1 dx}}{y_1^2}$$

and so

$$u = \int \frac{e^{-\int p_1 dx}}{y_1^2} dx$$

Example

Find the solution for

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0$$

Solution

We can spot that $y = x$ is a solution and so we let $y_1 = x$.
Now we will try to use this to obtain a second solution. First we rewrite the equation given as:

$$\frac{d^2y}{dx^2} - \frac{x(x+2)}{x^2}\frac{dy}{dx} + \frac{(x+2)}{x^2}y = 0$$

Then we can see that:

$$\begin{aligned} p_1 &= -\frac{x+2}{x} \\ &= -1 - \frac{2}{x} \end{aligned}$$

Therefore

$$\int p_1 dx = -x - 2\ln x + c$$

So

$$\begin{aligned} e^{-\int p_1 dx} &= e^{(x+2\ln x+c)} \\ &= de^x x^2 \end{aligned}$$

We are now in a position to find u

$$\begin{aligned} u &= \int \frac{e^{-\int p_1 dx}}{y_1^2} dx \\ &= d\int \frac{x^2 e^x}{x^2} dx \\ &= d\int e^x dx \\ &= de^x + e \end{aligned}$$

Therefore $y_2 = uy_1 = (de^x + e)x$.

Then the general solution to the homogeneous equation is

$$y = A'x + Bxe^x$$

and so $y_2 = xe^x$

Obtaining a Second Solution via the Wronskian

Consider

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$$

which has two solutions y_1 and y_2 and let $W(x) = y_1y_2' - y_2y_1'$. Then

$$\begin{aligned} \frac{dW}{dx} &= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= y_1y_2'' - y_1''y_2 \end{aligned}$$

However $y_i'' + p_1y_i' + p_2y_i = 0$.

$$\begin{aligned} \frac{dW}{dx} &= y_1(-p_1y_2' - p_2y_2) - y_2(-p_1y_1' - p_2y_1) \\ &= -p_1y_1y_2' - p_2y_1y_2 + p_1y_2y_1' + p_2y_1y_2 \\ &= -p_1W(x) \end{aligned}$$

$$\frac{W'}{W} = -p_1$$

$$\ln W = \int -p_1 dx$$

$$W = e^{\int -p_1 dx}$$

Using the definition of W:

$$y_1y_2' - y_2y_1' = e^{\int -p_1 dx}$$

$$\frac{y_2'}{y_1} - \frac{y_2y_1'}{y_1^2} = \frac{e^{\int -p_1 dx}}{y_1^2}$$

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{e^{\int -p_1 dx}}{y_1^2}$$

$$\frac{y_2}{y_1} = \int \frac{e^{\int -p_1 dx}}{y_1^2} dx$$

From which we can determine y_2 .

Example

Solve

$$xy'' - xy' + y = x^2$$

Solution

First consider

$$xy'' - xy' + y = 0$$

One obvious solution for this equation is $y_1 = x$. Seek a second solution by looking for $y_2 = xu(x)$.

$$x(xu'' + 2u') - x(xu' + u) + xu = 0$$

$$x^2u'' + (2x - x^2)u' + (-x + x)u = 0$$

Let $v = u'$

$$\begin{aligned} \frac{v'}{v} &= -\frac{2x - x^2}{x^2} \\ &= -\frac{2}{x} + 1 \end{aligned}$$

$$\ln v = -\ln x^2 + x + c$$

$$u' = a\frac{e^x}{x^2}$$

$$u = a\int \frac{e^x}{x^2} dx$$

So then we can write the solution to the homogeneous equation as:

$$y = Ax + Bx\int \frac{e^x}{x^2} dx$$

To find the particular integral we rewrite the equation as:

$$y'' - y' + \frac{1}{x}y = x$$

Use

$$y = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 \\ &= x\left[\int \frac{e^x}{x^2} dx + \frac{xe^x}{x^2}\right] - 1\left[x \int \frac{e^x}{x^2} dx\right] \\ &= \frac{x^2 e^x}{x^2} \\ &= e^x \end{aligned}$$

Now we need to evaluate

$$u_1 = -\int \frac{y_2 R}{W} dx$$

and

$$u_2 = \int \frac{y_1 R}{W} dx$$

Start with u_2

$$\begin{aligned} u_2 &= \int \frac{y_1 R}{W} dx \\ &= \int \frac{x^2}{e^x} dx \\ &= \int x^2 e^{-x} dx \\ &= \left[-x^2 e^{-x} + \int 2xe^{-x} dx\right] \\ &= \left[-x^2 e^{-x} - 2xe^{-x} + \int 2e^{-x} dx\right] \\ &= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \\ &= -(x^2 + 2x + 2)e^{-x} \end{aligned}$$

$$\begin{aligned} u_1 &= -\int \frac{y_2 R}{W} dx \\ &= -\int \frac{x^2 \int \frac{e^x}{x^2} dx}{e^x} dx \\ &= -\int (x^2 e^{-x} \int \frac{e^x}{x^2} dx) dx \\ &= +(x^2 + 2x + 2)e^{-x} \int \frac{e^x}{x^2} dx - \int (x^2 + 2x + 2) \frac{e^{-x} e^x}{x^2} dx \\ &= +(x^2 + 2x + 2)e^{-x} \int \frac{e^x}{x^2} dx - \int 1 + \frac{2}{x} + \frac{2}{x^2} dx \\ &= +(x^2 + 2x + 2)e^{-x} \int \frac{e^x}{x^2} dx - x - 2\ln x + \frac{2}{x} \end{aligned}$$

Now

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx \\ &= -x^2 - 2x \ln x + 2 \end{aligned}$$

Therefore

$$y = Ax + Bx \int \frac{e^x}{x^2} dx - x^2 - 2x \ln x + 2$$

Solution by Power Series

Consider

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2 y = R(x)$$

and a point of interest x_0.
Seek solution in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Example

$$(1 - x^2)\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 3y = 0$$

and $x_0 = 0$

Solution

We seek a solution of the form:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x)^n \\ &= y(0) + y'(0)x + y''\frac{x^2}{2!} + \ldots \end{aligned}$$

Let $x = 0$ then the equation yields

$$y''(0) = 3y(0)$$

Now differentiating our equation once we get

$$(1-x^2)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - 5x\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3\frac{dy}{dx} = 0$$

Let $x = 0$ and we obtain

$$y'''(0) = 8y'(0)$$

If we differentiate the equation given n times we obtain

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2xn\frac{d^{n+1}y}{dx^{n+1}} - (n^2-n)\frac{d^ny}{dx^n} - 5x\frac{d^{n+1}y}{dx^{n+1}} - 5n\frac{d^ny}{dx^n} - 3\frac{d^ny}{dx^n} = 0$$

Substitute $x = 0$ and we obtain

$$y^{n+2}(0) + [-n(n-1) - 5n - 3]y^n(0) = 0$$

$$y^{n+2}(0) - [n^2 + 4n + 3]y^n(0) = 0$$

$$y^{n+2}(0) = (n+1)(n+3)y^n(0)$$

So our expansion can we written as

$$y(x) = y(0) + y'(0)x + \frac{3}{2}y(0)x^2 + \frac{8}{3!}y'(0)x^3 + \ldots\ldots$$

where we have used:

$$y''(0) = 3y(0)$$

$$y'''(0) = 8y'(0)$$

$$\begin{aligned} y''''(0) &= 15y''(0) \\ &= 45y(0) \end{aligned}$$

$$\begin{aligned} y'''''(0) &= 24y'''(0) \\ &= 196y'(0) \end{aligned}$$

etc.

This can be written as:

$$y(x) = y(0)\left[1 + \frac{3}{2}x^2 + \frac{45}{4!}x^4 \ldots\ldots\right] + y'(0)\left[x + \frac{8}{3!}x^3 + \frac{196}{5!}x^5 \ldots\ldots\right]$$

Example

Solve

$$xy'' + (1+x)y' + 2y = 0$$

where $x_0 = 0$

Solution

Differentiating n times to get

$$xy^{n+2} + ny^{n+1} + (1+x)y^{n+1} + ny^n + 2y^n = 0$$

Setting $x = 0$ we obtain:

$$(n+1)y^{n+1}(0) + (n+2)y^n(0) = 0$$

$$y^{n+1} = -\frac{n+2}{n+1}y^n(0)$$

So

$$y'(0) = -2y(0)$$

$$\begin{aligned} y''(0) &= -\frac{3}{2}y'(0) \\ &= 3y(0) \end{aligned}$$

$$\begin{aligned} y'''(0) &= -\frac{4}{3}y''(0) \\ &= -4y(0) \end{aligned}$$

Therefore

$$\begin{aligned} y(x) &= y(0) - 2y(0)x + 3y(0)\frac{x^2}{2} - 4y(0)\frac{x^3}{3!} + \ldots\ldots \\ &= y(0)\left[1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \frac{5}{4!}x^4 + \ldots\ldots\right] \end{aligned}$$

Note that $1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \frac{5}{4!}x^4 - \ldots\ldots$ reminds us of $e^{-x} = [1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \ldots\ldots]$

So what happens if we integrate our expression for y.

$$\begin{aligned} \int y(x)dx &= y(0)x\left[1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \ldots\ldots\right] \\ &= y(0)xe^{-x} \end{aligned}$$

Now differentiate again

$$y(x) = y(0)[1-x]e^{-x}$$