AS2051

Section 2: Lagrange Multipliers

How do we maximize/minimize a function, say, $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$?

When we were considering the same problem but without a constraint we sought maxima and minima by considering

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

In this new type of problem, we form a new function L, which is called the Lagrange function.

$$L = f - \lambda g$$

where λ is the Lagrange multiplier. We then consider:

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \lambda} = 0$$

Note that $\frac{\partial L}{\partial \lambda} = 0$ simply gives us our constraint again.

In general we set

$$\frac{\partial L}{\partial x_1} = 0, \ldots, \frac{\partial L}{\partial x_i} = 0, \frac{\partial L}{\partial \lambda} = 0$$

Example

Maximize

$$F(x, y) = 2x^4 + 8x^2y^2 - 4(x^2 - y^2) + 2$$

on the curve

$$x^2 + y^2 - 4 = 0$$

Solution

For this example

$$L = 2x^4 + 8x^2y^2 - 4(x^2 - y^2) + 2 - \lambda(x^2 + y^2 - 4)$$

and so

$$\frac{\partial L}{\partial x} = 8x^3 + 16xy^2 - 8x - 2x\lambda$$

$$\frac{\partial L}{\partial y} = 16x^2y + 8y - 2y\lambda$$

$$\frac{\partial L}{\partial \lambda} = -x^2 - y^2 + 4$$

For stationary points we need:

$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial y} = 0$$

gives

$$(16x^2 + 8 - 2\lambda)y = 0$$

and

$$\frac{\partial L}{\partial x} = 0$$

gives

$$(8x^2 + 16y^2 - 8 - 2\lambda)x = 0$$

Therefore one possible point appears to be $x = 0, y = 0$. However, the point needs to satisfy $x^2 + y^2 = 4$. If we substitute $x = 0, y = 0$ into this equation we see that the left-hand side of the equation is 0 while the right hand side is 4. Therefore we need to look at other possible points.

- Consider $x = 0$ and now consider $y \neq 0$. If $y \neq 0$ then from $\frac{\partial L}{\partial y} = 0$ i.e. $(16x^2 + 8 - 2\lambda)y = 0$ we see that $8 - 2\lambda = 0$ and $x^2 + y^2 = 4$ gives $y = \pm 2$. ($\lambda = 4$).

- Consider now $y = 0$ and $x \neq 0$. If $x \neq 0$ then from $\frac{\partial L}{\partial x} = 0$ i.e. $(8x^2 + 16y^2 - 8 - 2\lambda)x = 0$ we see that $8x^2 - 8 - 2\lambda = 0$ and $x^2 + y^2 = 4$ gives $x = \pm 2$. ($\lambda = 12$).

- Consider now $x \neq 0$ and $y \neq 0$ then in this case we obtain the equations:

$$16x^2 + 8 - 2\lambda = 0$$

$$8x^2 + 16y^2 - 8 - 2\lambda = 0$$

$$x^2 + y^2 = 4$$

If we eliminate λ between the first two of these equations we obtain:

$$8x^2 - 16y^2 + 16 - 0$$

Now note that $y^2 = 4 - x^2$. Substitute this into $8x^2 - 16y^2 + 16 = 0$ and we obtain:

$$\begin{aligned} 8x^2 - 16(4 - x^2) + 16 &= 0 \\ 24x^2 - 48 &= 0 \\ x &= \pm\sqrt{2} \end{aligned}$$

Therefore from $y^2 = 4 - x^2$ we see that $y = \pm\sqrt{2}$.

Therefore, the stationary points are $(\pm 2, 0)$,$(0, \pm 2)$,$(\pm\sqrt{2}, \pm\sqrt{2})$.

Note that while we have found the stationary points it can often be difficult to determine the nature of the points. We can evaluate f at each of these points and we learn that:

$$f(\pm 2, 0) = 26$$

$$f(0, \pm 2) = 18$$

$$f(\pm\sqrt{2}, \pm\sqrt{2}) = 42$$

$(\pm 2, 0)$ is a local minima, $(0, \pm 2)$ global minima, $(\pm\sqrt{2}, \pm\sqrt{2})$ is a global maxima.

Example

A box has sides x, y, z. If it has volume V find x, y, and z so as to minimize the area.

Solution

Area is $2xy + 2yz + 2zx$ with volume $V = xyz$ and so

$$L = 2xy + 2yz + 2zx - \lambda(xyz - V)$$

Therefore,

$$\frac{\partial L}{\partial x} = 2y + 2z - \lambda yz$$

$$\frac{\partial L}{\partial y} = 2x + 2z - \lambda xz$$

$$\frac{\partial L}{\partial z} = 2x + 2y - \lambda xy$$

Recall that for a stationary point we require:

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \lambda} = 0$$

and so we have the following set of equations:

$$2y + 2z - \lambda yz = 0$$

$$2x + 2z - \lambda xz = 0$$

$$2x + 2y - \lambda xy = 0$$

If we multiply the first of these equations by x, the second of these equations by y and the third of these equations by z we obtain:

$$2yx + 2zx - \lambda V = 0$$

$$2xy + 2zy - \lambda V = 0$$

$$2xz + 2yz - \lambda V = 0$$

Now subtracting the second equation $(2xy + 2zy - \lambda V)$ from the first of these equations $(2yx + 2zx - \lambda V)$ we obtain:

$$\begin{aligned} 2xz - 2yz &= 0 \\ 2z(x - y) &= 0 \end{aligned}$$

From this we deduce that $x = y$ as $z \neq 0$. Similarly we subtract the third of our equations $(2xz + 2yz - \lambda V)$ from the second of our equations $(2xy + 2zy - \lambda V)$ to obtain:

$$\begin{aligned} 2xy - 2xz &= 0 \\ 2x(y - z) &= 0 \end{aligned}$$

From this we deduce that $y = z$ as $x \neq 0$. So we have $x = y = z$. If we substitute in to $xyz = V$ we can, for example, obtain $x^3 = V$ and so $x = V^{\frac{1}{3}} = y = z$. So the smallest area is that of a cube!

Example

Maximize $f(x, y) = 25 - x^2 - y^2$ subject to $2x + y = 4$

Solution

We begin by forming:

$$L = 25 - x^2 - y^2 - \lambda(2x + y - 4)$$

and so:

$$\frac{\partial L}{\partial x} = -2x - 2\lambda$$

$$\frac{\partial L}{\partial y} = -2y - \lambda$$

$$\frac{\partial L}{\partial \lambda} = -2x - y + 4$$

Now recall that for a stationary point here we require:

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

and so we have the set of equations:

$$-2x - 2\lambda = 0$$

$$-2y - \lambda = 0$$

$$-2x - y + 4 = 0$$

Subtracting twice the second equation ($2y - \lambda = 0$) from the first equation ($-2x - 2\lambda = 0$) we obtain:

$$\begin{aligned} -2x + 4y &= 0 \\ x &= 2y \end{aligned}$$

If we substitute $x = 2y$ into $-2x - y + 4 = 0$ we obtain:

$$-5y + 4 = 0$$

and so there is one point $y = 4/5$, $x = 8/5$. For this point $f = 25 - \frac{16}{25} - \frac{64}{25}$ i.e. $f = \frac{545}{25}$.

Bordered Hessian Test

For two variables and one constraint.

$$\begin{pmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{pmatrix}$$

- If $|H_3| > 0$, evaluated at the point, then the point is a maximum.
- If $|H_3| < 0$, evaluated at the point, then the point is a minimum

For our example

$$\begin{pmatrix} 0 & -2 & -1 \\ -2 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$$

and so $|H_3| = 10$ and so it is a maximum.

Example

Find the maxima and minima for

$$f = 2(2y - z)$$

subject to

$$2x - 2y - z = 2$$

$$x^2 + y^2 = 3$$

$$L = 2(2y - z) - \lambda(2x - 2y - z - 2) - \mu(x^2 + y^2 - 3)$$

$$\frac{\partial L}{\partial x} = -2\lambda - 2x\mu$$

$$\frac{\partial L}{\partial y} = 4 + 2\lambda - 2y\mu$$

$$\frac{\partial L}{\partial z} = -2 + \lambda$$

$$\frac{\partial L}{\partial \lambda} = -(2x - 2y - z - 2)$$

$$\frac{\partial L}{\partial \mu} = -(x^2 - y^2 - 3)$$

Therefore we have

$$-2\lambda - 2x\mu = 0$$

$$4 + 2\lambda - 2y\mu = 0$$

$$-2 + \lambda = 0$$

$$2x - y - z = 2$$

$$x^2 + y^2 = 3$$

Now from these equations we can determine $\lambda = 2$. We substitute this into the first two equations and we get:

$$0 = -4 - 2x\mu$$
$$0 = 8 - 2y\mu$$

and so we deduce

$$x = -\frac{2}{\mu}$$

$$y = \frac{4}{\mu}$$

Substituting each of these onto $x^2 + y^2 = 3$ we can obtain:

$$\begin{aligned} \frac{4}{\mu^2} + \frac{16}{\mu^2} &= 3 \\ \frac{20}{\mu^2} &= 3 \\ \mu^2 &= \frac{20}{3} \\ \mu &= \pm\sqrt{\frac{20}{3}} \end{aligned}$$

If $\mu = \sqrt{\frac{20}{3}}$ then $x = -2/\sqrt{\frac{20}{3}}$,$y = 4/\sqrt{\frac{20}{3}}$, $z = -12/\sqrt{\frac{20}{3}} - 2$ and for this $f = 16/(\sqrt{\frac{20}{3}}) - 2(-12/\sqrt{\frac{20}{3}} - 2)$. This is a maximum.
If $\mu = -\sqrt{\frac{20}{3}}$ then $x = 2/\sqrt{\frac{20}{3}}$,$y = -4/\sqrt{\frac{20}{3}}$, $z = 12/\sqrt{\frac{20}{3}} - 2$ and for this $f = -16/(\sqrt{\frac{20}{3}}) - 2(12/\sqrt{\frac{20}{3}} - 2)$. This is a minimum.