Section A: Calculus

1. (a) Sketch the region of integration in the double integral

$$I = \int_{y=0}^{y=2} dy \int_{x=y/2}^{x=1} \cos(x^2) dx.$$

By changing the order of integration, evaluate I.

(b) The cylindrical coordinates (r, θ, z) are related to the Cartesian coordinates (x, y, z) by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Obtain the Jacobian determinant of the transformation from Cartesian to cylindrical coordinates. Hence use cylindrical coordinates to compute the integral

$$V = \int\int\int_R (x^2 + y^2)^2 dx\, dy\, dz$$

on a region R corresponding to a circular cylinder of radius 1 centered at the origin and located above the $z = 1$ plane and below the $z = 5$ plane.

2. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function

$$f(x, y) = xe^{-x^2+y^2}.$$

(b) Compute the Taylor's expansion of the function

$$f(x, y) = x^2 + xy + y^3.$$

around the point $(0, 1)$ including up to second order terms. Hence estimate the value of the function at the point $(0.1, 1.1)$.

3. Consider the following transformation of coordinates

$$x = e^{u+v}, \qquad y = e^{u-v}$$

in two dimensional space. Let $f(x, y)$ be a twice differentiable function of x and y.

Turn over ...

(a) Use the chain rule to express the first order partial derivatives of f with respect to u and v in terms of the first order partial derivatives of f with respect to x and y.

(b) Likewise, use the chain rule to find $\partial^2 f/\partial u^2$ and $\partial^2 f/\partial v^2$. Hence show that

$$\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = 2e^{2u}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right).$$

4. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1 u_1(x) + c_2 u_2(x)$ is the general solution of the following homogeneous second-order differential equation

$$y'' - 4y = 0,$$

where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation

$$y'' - 4y = \cosh(2x).$$

Hence determine the general solution of this inhomogeneous equation.

Section B: Linear Algebra

In the following questions, $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and all polynomials of degree at most n with real coefficients respectively.

5. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $U = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) \mid a+b+c+d=0\} \subset M(2,2)$.

   ii. $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1+x_2)^2 = x_3\} \subset \mathbb{R}^3$.

   iii. $W = \{a_0 + a_1x + a_2x^2 \in P_2 \mid a_0 = a_1 + 2a_2\} \subset P_2$

   (b) Find a basis for the real vector space P_n. What is the dimension of P_n?

   (c) Do the following sets form a basis for $\mathbb{R}^3$? If not, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   i. $\{(1,5,-2), (-2,1,1), (0,0,3)\}$

   ii. $\{(1,0,0), (2,1,0), (3,2,1), (-1,-1,-1)\}$

6. (a) Show that the map $f : P_2 \to \mathbb{R}^4$ given by

$$f(a_0 + a_1x + a_2x^2) = (a_1 + a_0, a_2 + a_0, a_1, a_2)$$

   is linear. Write down the matrix representing f with respect to the standard ordered basis $\{1, x, x^2\}$ of P_2 and the standard ordered basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ of $\mathbb{R}^4$.

   (b) Is there a linear map $f : \mathbb{R}^3 \to P_1$ such that $f(1,0,0) = x$, $f(1,1,1) = 1 + x$ and $f(2,0,0) = 3$? Justify your answer.

   (c) Define the image, the kernel, the rank and the nullity of a linear map and state carefully the Rank-Nullity theorem.

   (d) Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x,y,z) = (x+y+z, z, z).$$

   Determine whether f is injective, surjective, both or neither and find a basis for the image of f and a basis for the kernel of f.

Turn over ...

7. (a) Let A be a real $n \times n$ matrix. Define what is meant by an eigenvector and an eigenvalue for A.

   (b) State the diagonalization theorem for matrices.

   (c) Let $A = \begin{pmatrix} 7 & 4 & -8 \\ 8 & 3 & -8 \\ 8 & 4 & -9 \end{pmatrix}$. Use the diagonalization theorem to find an invertible 3×3 matrix P and its inverse P^{-1} such that $P^{-1}AP$ is diagonal.

8. Consider the real vector space $\mathbb{R}^4$ with real inner product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$.

   (a) Define the norm of a vector $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with respect to the above inner product. What is the norm of $(-1, 2, -5, 3)$?

   (b) When do we say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ are orthogonal? Are $(1, 3, 0, 2)$ and $(-2, 0, 7, -1)$ orthogonal?

   (c) What is an orthonormal set of vectors in $\mathbb{R}^4$?
   Is $\{(1, 0, 0, 0), (0, \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}), (0, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}})\}$ an orthonormal set? Justify your answer.

   (d) Use the Gram-Schmidt process to construct an orthonormal basis for $\mathbb{R}^4$ starting from the basis

$$\{(0, 0, 0, 1), (1, 0, 1, 1), (1, 1, 1, 0), (0, -1, 1, 0)\}$$

Internal Examiners:	Dr O. Castro Alvaredo
	Dr M. De Visscher
External Examiners:	Professor M.E. O'Neill
	Professor J. Billingham