Linear Algebra Coursework 1

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the mathematics general office (CM520) by **4:00pm on Monday 12th November**. Late submissions will be penalised.

1. For each of the following subsets of $\mathbb{R}^n$ (with $n \geq 3$) either prove that it is a subspace or show that one of the subspace axioms fails.

   (a) $\{(x_1, x_2, \ldots, x_n) : x_1 + x_2 = 0\}$.

   (b) $\{(x_1, x_2, \ldots, x_n) : x_1^2 - x_2 = 0\}$.

   (c) $\{(x_1, x_2, \ldots, x_n) : x_1 - x_2 + x_3 \geq 0\}$.

   (d) $\{(x_1, x_2, \ldots, x_n) : x_1 = 0 \text{ or } x_2 = 0\}$.

   [20]

2. For each of the following sets, either prove or disprove that it is a basis for P_2 (you should state clearly any theorems or other standard results that you use). For those sets which are not bases, determine whether they are linearly independent, a spanning set, or neither.

   (a) $\{1 + x, x + x^2\}$.

   (b) $\{2x^2 - 1, 1 + 3x - 4x^2, 1 + x + x^2\}$.

   (c) $\{1 + 2x + x^2, 1 - x - 4x^2, x - x^2, 1 + 3x\}$.

   (d) $\{3 + 4x + x^2, 1 + x + x^2, 7 + 10x + x^2\}$.

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3. For each of the following maps, either prove the map is linear or give an example to show that linearity fails.

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (2x - y - z, x + z, y)$.

   (b) $f : M(2, 2) \to \mathbb{R}^2$, with $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a - c + d + 1, \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix})$.

   (c) $f : P_2 \to P_2$, with $(f(p))(x) = p(x + 1) + 2\frac{d}{dx}p(x)$.

   (d) $f : \mathbb{C}^2 \to \mathbb{C}^2$, regarded as complex vector spaces, with $f(z_1, z_2) = (\bar{z}_2, \bar{z}_1 + z_2)$ (where $\bar{z}$ is the complex conjugate of z).

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4. For each of the following linear maps find a basis for the image and the kernel (you should state clearly any theorems or other standard results that you use).

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (2x - y, z, 2x + 2z - y)$.

   (b) $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (x + 2y - z, z - x - 2y, 0)$.

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5. Write down each of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ of $\mathbb{R}^3$ in coordinate form with respect to the basis $\{\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3\}$.

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6. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map given on the standard basis by $f(\mathbf{e}_1) = \mathbf{e}_2 + 3\mathbf{e}_3$ and $f(\mathbf{e}_2) = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 5\mathbf{e}_3$. Write down the matrix for this map with respect to the bases $\{2\mathbf{e}_1 + 3\mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}$ of $\mathbb{R}^2$ and $\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ of $\mathbb{R}^3$.

   [15]