Linear Algebra: Coursework 1

This is an assessed coursework. Solutions should be handed in to the **mathematics general office** (CM326) by **4pm on Thursday 2nd November 2006**. Late submissions will be penalised.

• $M(2,2)$ denotes the vector space over $\mathbb{R}$ consisting of all 2×2 matrices with real entries (with the usual addition and scalar multiplication of matrices).

• P_n (for $n \geq 1$) denotes the vector space over $\mathbb{R}$ of all polynomials in one variable with real coefficients of degree at most n (with the usual addition and scalar multiplication of polynomials).

• $\mathbb{R}^{\mathbb{R}}$ denotes the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ (with the usual addition and scalar multiplication of functions).

1. (a) Is $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_4 = 0\}$ a subspace of $\mathbb{R}^4$? If it is, prove it and if it is not, show that one of the subspace conditions fails.

   (b) Is $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) \mid a + b = c + d\}$ a subspace of $M(2,2)$? If it is, prove it and if it is not, show that one of the subspace conditions fails.

   (c) Is $\{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = 1\}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? If it is, prove it and if it is not, show that one of the subspace conditions fails.

   [15]

2. Let V be any vector space over $\mathbb{R}$ and let U and W be any subspaces of V.

   (a) Is it always true that $U \cap W$ is a subspace of V? If it is, prove it. If it is not, find examples of such V, U and W where $U \cap W$ is not a subspace of V.

   (b) Is it always true that $U \cup W$ is a subspace of V? If it is, prove it. If it is not, find examples of such V, U and W where $U \cup W$ is not a subspace of V.

   [5]

3. For each of the following sets, either prove or disprove that it is a basis for V. (Clearly state any standard result that you use). For those sets which are not a basis, determine whether they are linearly independent, spanning or neither.

   (a) $\{(0,0,1), (1,0,1), (0,1,0), (-1,-1,0)\}$ for $V = \mathbb{R}^3$.

   (b) $\{(0,0,0,1), (3,0,1,0), (5,4,3,-2)\}$ for $V = \mathbb{R}^4$.

   (c) $\{3, 2 - x, 4 + x - x^2\}$ for $V = P_2$.

   [15]

4. Determine whether the following maps are linear. Justify your answers.

   (a) $f : P_n \longrightarrow P_{n+1}$ defined by $f(p(x)) = x^2 \frac{d}{dx}(p(x))$ for all polynomials $p(x) \in P_n$.

   (b) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $f(x,y,z) = (x+y)z$ for all $(x,y,z) \in \mathbb{R}^3$.

   [10]

5. Is there a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that

   (a) $f(1,1) = (1,0,0)$, $f(2,0) = (1,2,3)$ and $f(4,2) = (0,0,-5)$? If there isn't one, explain why not. If there is one, find $f(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

   (b) $f(1,1) = (1,0,0)$, $f(2,0) = (1,2,3)$ and $f(4,2) = (3,2,3)$? If there isn't one, explain why not. If there is one, find $f(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

   [5]