Linear Algebra Coursework 2

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the mathematics general office (CM520) by **4:00pm on Monday 21st January**. Late submissions will be penalised.

1. Find the eigenvalues of
$$\begin{pmatrix} -17 & 0 & 8 \\ -6 & 1 & 2 \\ -48 & 0 & 23 \end{pmatrix}.$$
For each eigenvalue find a basis for the corresponding eigenspace.

[15]

2. Let A be the matrix
$$\begin{pmatrix} 5 & -2 & -2 \\ 0 & 3 & 0 \\ 4 & -4 & -1 \end{pmatrix}.$$
By finding a basis of eigenvectors, determine invertible matrices P and P^{-1} such that the matrix $P^{-1}AP$ is diagonal. Hence calculate the value of A^{10}.

[20]

3. For each of the following functions, determine whether they give a real inner product. Give reasons for your answers.
   (a) $\langle -,- \rangle$ on $\mathbb{R}^3$ given by $\langle \mathbf{x},\mathbf{y} \rangle = x_1y_1 + x_3y_3$.
   (b) $\langle -,- \rangle$ on $\mathbb{R}^3$ given by $\langle \mathbf{x},\mathbf{y} \rangle = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$.
   (c) $\langle -,- \rangle$ on $M(2,2)$ given by $\langle A,B \rangle = \det(BA)$.
   (d) $\langle -,- \rangle$ on $M(2,2)$ given by $\langle A,B \rangle = \mathrm{tr}(AB)$.
   (e) $\langle -,- \rangle$ on P_2 given by $\langle p,q \rangle = p(0)q(0) + p(\frac{1}{2})q(\frac{1}{2}) + p(1)q(1)$.

[15]

4. Verify that the set
$$\{(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{-1}{\sqrt{15}}, \frac{3}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{-1}{\sqrt{15}}), (\frac{3}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{-1}{\sqrt{15}}, \frac{-2}{\sqrt{15}}), (0, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$$
of elements from $\mathbb{R}^4$ form an orthonormal basis with respect to the usual scalar (i.e. "dot") product $\mathbf{x}.\mathbf{y}$. Write the element $(2,-3,4,-5)$ as a linear combination of these basis elements.

[10]

[Continued overleaf]

5. Use the Gram-Schmidt method to form an orthonormal basis for $M(2,2)$ with inner product $\langle A,B \rangle = \mathrm{tr}(B^TA)$ from the set of basis elements
$$\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}.$$

[20]

6. Let A be the matrix
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$
By finding a *suitable* basis of eigenvectors, determine *orthogonal* matrices P and P^T such that the matrix P^TAP is diagonal.

[20]