Linear Algebra: Coursework 2

This is an assessed coursework. Solutions should be handed in to the **mathematics general office** (CM326) by **4pm on Thursday 30th November**. Late submissions will be penalised.

• P_n denotes the vector space over $\mathbb{R}$ of all polynomials in one variable with real coefficients of degree at most n (with the usual addition and scalar multiplication of polynomials).

1. Consider the linear map

$$f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2 \; : \; (x, y, z, t) \mapsto (x + y, 2z + 2t)$$

for all $(x, y, z, t) \in \mathbb{R}^4$. Determine whether f is injective, surjective, both or neither. Moreover, find bases for the kernel and the image of f. (You may use the Rank-Nullity Theorem).

[10]

2. Consider the linear map

$$g : P_2 \to P_2 \; : \; p(x) \mapsto p(1) + \frac{d}{dx}(p(x))$$

for all $p(x) \in P_2$.

(a) Find $g(1), g(x)$ and $g(x^2)$ and hence find the matrix A representing g with respect to the ordered basis $\{1, x, x^2\}$ of P_2.

(b) Find $g(1 + x)$ and $g(1 + x + x^2)$ and hence find the matrix B representing g with respect to the ordered basis $\{1, 1 + x, 1 + x + x^2\}$ of P_2.

(c) Express each vector $1, 1 + x$ and $1 + x + x^2$ in coordinate form with respect to the ordered basis $\{1, x, x^2\}$ and hence find the change of basis matrix P and check that it satisfies the Change of basis Theorem, namely

$$P^{-1}AP = B.$$

[10]

3. Show that the matrix $A = \begin{pmatrix} 2 & 0 & -3 \\ -6 & -1 & 6 \\ 0 & 0 & -1 \end{pmatrix}$ is diagonalizable and hence find a matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

[25]

4. (a) Let $h_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear map with $\operatorname{rank} h_1 = 3$. Determine whether h_1 is surjective, injective, both or neither. Justify your answer.

(b) Let $h_2 : V \longrightarrow V$ be a linear map from a finite-dimensional vector space V to itself. Assume that 0 is an eigenvalue for the map h_2. Determine whether h_2 is injective, surjective, both or neither. Justify your answer.

(You may use any standard result seen at the lecture provided you state them clearly). **[5]**