Section B: Linear Algebra

In the following questions, $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and all polynomials of degree at most n with real coefficients respectively.

1. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - x_4 = 0\}$ in $\mathbb{R}^4$

   ii. $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ab = 1\}$ in $M(2,2)$

   iii. $\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in P_n \mid a_0 = 0\}$ in P_n

   (b) Find two different bases for the real vector space $\mathbb{R}^2$.

   (c) Do the following sets form a basis for $\mathbb{R}^3$? If not, determine whether they are linearly independent, a spanning set for $\mathbb{R}^3$, or neither.

   i. $\{(0,0,1), (0,2,1), (3,2,1), (4,5,6)\}$

   ii. $\{(1,0,1), (4,3,-1), (5,3,-4)\}$

2. (a) Are the following maps linear? Justify your answers.

   i. $f : P_2 \to \mathbb{R}^3 : a_0 + a_1 x + a_2 x^2 \mapsto (a_0, a_1, a_2)$

   ii. $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + y, y^2)$

   iii. $f : M(2,2) \to M(2,2) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}$

   (b) Is there a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(1,1) = (1,2,3)$, $f(2,1) = (0,0,1)$ and $f(2,2) = (0,1,2)$? Justify your answer.

   (c) Define the rank and the nullity of a linear map and state carefully the Rank-Nullity theorem.

   (d) Find bases for the image and the kernel of the linear map given by

   $$f : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x + y, 0, y + z)$$

Turn over ...

3. (a) Let A be a real $n \times n$ matrix. Define what is meant by an eigenvector and an eigenvalue for A.

(b) State the diagonalization theorem for matrices.

(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Use the diagonalization theorem to find an invertible 3×3 matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

4. Consider the real vector space $\mathbb{R}^4$ with real inner product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$.

(a) Define the norm of a vector $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with respect to the above inner product. What is the norm of $(2, -3, 5, 1)$?

(b) When do we say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ are orthogonal? Are $(0, 1, -1, 1)$ and $(6, 3, 3, 27)$ orthogonal?

(c) What is an orthonormal set of vectors in $\mathbb{R}^4$?
Is $\{(0, 0, 1, 0), (\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}), (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0)\}$ an orthonormal set? Justify your answer.

(d) Use the Gram-Schmidt process to construct an orthonormal basis for $\mathbb{R}^4$ starting from the basis

$$\{(1, 0, 0, 0), (1, 1, 0, 1), (0, 1, 1, 1), (0, 1, -1, 0)\}$$

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