Section B: Linear Algebra

In the following questions, $M(2,2)$ and P_n denote the vector spaces over $\mathbb{R}$ of all real-valued 2×2 matrices and all polynomials of degree at most n with real coefficients respectively.

1. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $U = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) \mid a^2 = d^2\}$ in $M(2,2)$

   ii. $V = \{p(x) \in P_n \mid p(3) = 0\}$ in P_n

   iii. $W = \{(x, y, z, t) \in \mathbb{R}^4 \mid y = z + t\}$ in $\mathbb{R}^4$

   (b) Find a basis for the real vector space $\mathbb{R}^3$ containing the vector $(3, 5, -4)$.

   (c) Do the following sets form a basis for V? If not, determine whether they are linearly independent, a spanning set for V, or neither.

   i. $\{(1,0,1), (1,1,0), (0,1,1), (1,1,1)\}$ for $V = \mathbb{R}^3$.

   ii. $\{5, 2 + x - 3x^2, 4x - 1\}$ for $V = P_2$.

2. (a) Determine whether the following maps are linear or not. Justify your answers.

   i. $f : \mathbb{R}^2 \to P_5 : (a, b) \mapsto (a + b)x^5$.

   ii. $f : M(2,2) \to \mathbb{R}^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad, bc)$.

   (b) Let V, W be two real finite-dimensional vector spaces and let $f : V \to W$ be a linear map. Define what is meant by the image, the kernel, the rank and the nullity of f and state the Rank-Nullity theorem.

   (c) Let $f : P_2 \to \mathbb{R}^3$ be a linear map with nullity $f = 1$. Determine whether the map f is injective, surjective, both or neither. Justify your answer.

   (d) Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^3$ given by

   $$f(x, y, z, t) = (x + y, 0, z + t)$$

   for all $(x, y, z, t) \in \mathbb{R}^4$. Determine whether f is injective, surjective, both or neither and find a basis for the kernel of f and a basis for the image of f.

Turn over ...

3. (a) Define what is meant by an eigenvector and an eigenvalue for a real $n \times n$ matrix.

(b) Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$. Show that the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A. What is the corresponding eigenvalue?

(c) Show that the matrix A given in (b) is diagonalizable and hence find an invertible 3×3 matrix P (and P^{-1}) such that $P^{-1}AP$ is diagonal.

4. Consider the real vector space $M(2,2)$ with real inner product given by

$$\langle A, B \rangle = \mathrm{tr}(B^T A)$$

for all $A, B \in M(2,2)$.

(a) Define the norm of a matrix $A \in M(2,2)$ with respect to the above inner product. What is the norm of $\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}$?

(b) When do we say that two matrices $A, B \in M(2,2)$ are orthogonal (with respect to the above inner product)? Are $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix}$ orthogonal?

(c) What is an orthonormal set of matrices in $M(2,2)$ (with respect to the above inner product)? Is

$$\{A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, A_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$$

an orthonormal set? Justify your answer.

(d) Let $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using (c) and the fact that $\{A_1, A_2, A_3, B\}$ is a basis for $M(2,2)$, find a matrix A_4 such that $\{A_1, A_2, A_3, A_4\}$ is an orthonormal basis for $M(2,2)$.

(e) Find $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4.$$

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