Linear Algebra: Exercise Sheet 1

1. Show that P_n, the set of all polynomials in one variable with real coefficients, with the usual addition and scalar multiplication of polynomials, is a vector space over $\mathbb{R}$.

2. Show that $M(2,2)$, the set of all 2×2 matrices with real entries, with the usual addition and scalar multiplication of matrices, form a vector space over $\mathbb{R}$.

3. Which of the following subsets are subspaces of $\mathbb{R}^2$?

   (a) $\{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$
   (b) $\{(x,y) \in \mathbb{R}^2 \mid x, y \text{ are integers}\}$
   (c) $\{(x,y) \in \mathbb{R}^2 \mid x \leq y\}$
   (d) $\{(x,y) \in \mathbb{R}^2 \mid x^2 = y^2\}$
   (e) $\{(x,y) \in \mathbb{R}^2 \mid y = 2x\}$

4. Which of the following subsets are subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$?

   (a) $\{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = f(1)\}$
   (b) $\{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) \geq 0 \quad \forall x \in \mathbb{R}\}$
   (c) $\{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) = f(-x) \quad \forall x \in \mathbb{R}\}$ (i.e. the subset of all even functions)

5. Which of the following subsets are subspaces of P_n?

   (a) $\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in P_n \mid a_0 + a_1 + a_2 + \ldots + a_n = 0\}$
   (b) the set of all polynomials of degree exactly n.
   (c) the set of all polynomials $p(x) \in P_n$ satisfying $p(0) = 0$.

6. Which of the following subsets are subspaces of $M(2,2)$?

   (a) the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$
   (b) the set of all 2×2 matrices of the form $\begin{pmatrix} a & 1 \\ 0 & c \end{pmatrix}$ for some $a, c \in \mathbb{R}$
   (c) $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,2) \mid ad - cb = 4\}$

7. **(optional)** Is $\mathbb{C}^n$ a vector space over $\mathbb{R}$? Justify your answer.
   Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$? Justify your answer.