Linear Algebra: Exercise Sheet 3

1. For each of the following maps, either prove that it is linear or give an example to show where linearity fails.

   (a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R} \,:\, (x, y) \mapsto 3x + 2y$

   (b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \,:\, (x, y) \mapsto (xy, 0)$

   (c) $f : P_n \longrightarrow P_{n+1} \,:\, p(x) \mapsto (x + 1)p(x)$

   (d) $f : P_n \longrightarrow \mathbb{R} \,:\, p(x) \mapsto \int_0^1 p(x)dx$

   (e) $f : P_n \longrightarrow P_n \,:\, p(x) \mapsto \frac{d}{dx}(p(x)) + (5x + 2)$

   (f) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \,:\, (x, y, z) \mapsto (y + z, x + z, x + y)$

   (g) $f : M(n, m) \longrightarrow M(m, n) \,:\, A \mapsto A^T$

   (h) $f : M(n, n) \longrightarrow \mathbb{R} \,:\, A \mapsto \det(A)$

2. Is there a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $f(1, 0) = (3, 2, 1)$, $f(1, 1) = (-1, 0, 1)$ and $f(3, 1) = (5, 0, -2)$?

3. For each of the following linear maps, determine whether they are injective, surjective, both or neither.

   (a) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \,:\, (x, y, z) \mapsto (x + y, z)$

   (b) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \,:\, (x, y, z) \mapsto (y + z, x + z, x + y)$

   (c) $f : M(2, 2) \longrightarrow \mathbb{R} \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + b + c + d$

   (d) $f : P_1 \longrightarrow P_2 \,:\, a_0 + a_1x \mapsto a_0x + a_1x^2$

   (e) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \,:\, (x, y, z) \mapsto (y, y, y)$

   (f) $f : P_n \longrightarrow P_n \,:\, p(x) \mapsto p(x) - p(0)$

4. Find a basis for the image and the kernel of each of the linear maps in question 3 (a)–(f). (You may use the Rank-Nullity theorem).

5. **(optional)** Can you find linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

   (a) $\operatorname{Im} f \subsetneq \operatorname{Ker} f$?

   (b) $\operatorname{Ker} f \subsetneq \operatorname{Im} f$?

   (c) $\operatorname{Ker} f = \operatorname{Im} f$?

6. **(optional)** Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be linear maps between vector spaces over the same $\mathbb{F}$. Consider the composition $(g \circ f) : U \longrightarrow W$. Show that $\operatorname{Ker}(f) \subseteq \operatorname{Ker}(g \circ f)$. Deduce that $\operatorname{Rank}(g \circ f) \leq \operatorname{Rank}(f)$.