Linear Algebra: Exercise Sheet 4

1. Write each of the following vectors in coordinate form with the respect to the given ordered basis.

   (a) $7\mathbf{e_1} + 5\mathbf{e_2} - \mathbf{e_3}$; basis of $\mathbb{R}^3$: $\{\mathbf{e_1}, \mathbf{e_1} + \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}\}$

   (b) $5x^2 - 2x + 3$; basis of P_2: $\{1, 1 + x, 1 + x^2\}$

   (c) $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3}$; basis: $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_1} - \mathbf{v_2}, \mathbf{v_3}\}$, where $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis of a 3-dimensional vector space over $\mathbb{R}$.

2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map given by $f(\mathbf{e_1}) = 2\mathbf{e_1} - \mathbf{e_3}$ and $f(\mathbf{e_2}) = \mathbf{e_2} + \mathbf{e_3}$. Write down the matrix of f with respect to the basis of $\mathbb{R}^2$ given by $\{\mathbf{e_1} - \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2}\}$ and the basis of $\mathbb{R}^3$ given by $\{\mathbf{e_1}, \mathbf{e_1} + \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}\}$.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : (x, y) \mapsto (x, y, y)$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 : (x, y, z) \mapsto (x + y, z)$. Fix the basis $\{\mathbf{e_1} - \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2}\}$ for $\mathbb{R}^2$ and the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ for $\mathbb{R}^3$. Find the matrices of f and g with respect to these bases. Find also the matrix of the linear map $(g \circ f)$ in these bases. Check that the product rule holds in this case.

4. Let $f : P_2 \longrightarrow P_2 : p(x) \mapsto \frac{d}{dx}(p(x))$. Find the matrix of f with respect to the basis $\{1, x, x^2\}$ for both spaces. Find also the matrix of f with respect to another basis given by $\{1, 1 + x, 1 + x^2\}$. Verify that the change of basis theorem holds in this case.

5. **(optional)** Let f and g be linear maps from V to W, where V and W are vector spaces over $\mathbb{F}$. Define the map $(f + g) : V \longrightarrow W$ by setting $(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$ for all $\mathbf{v} \in V$. Check that $f + g$ is linear. For $\lambda \in \mathbb{F}$, define $(\lambda f) : V \longrightarrow W$ by setting $(\lambda f)(\mathbf{v}) = \lambda(f(\mathbf{v}))$. Check that (λf) is also a linear map.

   Now, fix bases for V and W and write A, resp. B, for the matrix representing f, resp. g, in these bases. Show that the matrix representing $(f + g)$ in these bases is given by $A + B$ and that the matrix representing (λf) in these bases is given by λA.