Linear Algebra: Exercise Sheet 5

1. Decide whether the following statements are true or false. Justify your answers.

   (a) If λ is an eigenvalue of an invertible linear map $f : V \longrightarrow V$, where V is a vector space over $\mathbb{F}$, then λ^{-1} is an eigenvalue of the linear map f^{-1}.

   (b) If λ is an eigenvalue of an $n \times n$ matrix A then λ^m is an eigenvalue of the matrix A^m.

2. For each of the following matrices A, find the eigenvalues over $\mathbb{R}$ and find a basis for each eigenspace. If the matrix A satisfies the conditions of the diagonalisation theorem, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

   (a) $\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} -2 & 0 \\ 6 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 0 & 0 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$

3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95 percent will be healthy tomorrow. Of the students who are ill today, 55 percent will be ill tomorrow. Let x_0 (respectively y_0) be the proportion of students who are healthy (respectively ill) on Monday. Let x_n (resp. y_n) be the proportion of students who are healthy (resp. ill) after n days.

   (a) Find a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$

   for all $n \geq 0$.

   (b) Suppose that 20 percent of students are ill on Monday. What percentage of students will still be ill on Tuesday? on Wednesday? in n days?
   What will be the situation in the long term (as n gets very large)?