Linear Algebra: Exercise Sheet 6

1. For each of the following functions, either prove that they give a real inner product or give an example to show how they fail.

   (a) $\langle -, - \rangle$ on $\mathbb{R}^3$ given by $\langle \mathbf{x}, \mathbf{y} \rangle = 3x_1y_1 + 2x_2y_2 + 4x_3y_3$.

   (b) $\langle -, - \rangle$ on $\mathbb{R}^3$ given by $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 6x_2y_2 - 6x_3y_3$.

   (c) $\langle -, - \rangle$ on $\mathbb{R}^2$ given by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1x_2y_1y_2$.

   (d) $\langle -, - \rangle$ on P_2 given by $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$.

   (e) $\langle -, - \rangle$ on P_2 given by $\langle p(x), q(x) \rangle = p_0q_0 + p_1q_1 + p_2q_2$, where $p(x) = p_0 + p_1x + p_2x^2$ and $q(x) = q_0 + q_1x + q_2x^2$.

2. Using the inner product on matrices given by $\langle A, B \rangle = tr(B^T A)$, calculate the inner product $\langle A, B \rangle$ and the norm of A for $A = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 3 \\ -1 & -2 & 6 \end{pmatrix}$.

3. Calculate the inner product and the norm of $x + 3$ and $x^2 + x - 1$ using each of the functions (d) and (e) of question 1 which is an inner product.

4. Which of the following sets are orthogonal? Which are orthonormal?

   (a) $\{(1, 1, 1, 1), (1, -1, 0, 0), (0, 0, 1, -1)\} \subset \mathbb{R}^4$ with the usual dot product.

   (b) $\{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ with the inner product given by $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 4x_2y_2$.

   (c) $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix} \right\} \subset M(2, 2)$ with the inner product on M(2,2) given by $\langle A, B \rangle = tr(B^T A)$.

   (d) $\{x, x^2, 1 - \frac{5}{3}x^2\} \subset P_2$ with inner product given by $\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx$.

   (e) $\{1, 1+x, 1+x+x^2\} \subset P_2$ with inner product given by $\langle p(x), q(x) \rangle = p_0q_0 + p_1q_1 + p_2q_2$ (as in question 1(e)).

5. **(Optional)** Prove the generalized Pythagoras Theorem which is stated as follows. Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_r}\}$ is an orthogonal set of vectors in a real inner product space, then we have

$$||\mathbf{v_1} + \mathbf{v_2} + \ldots + \mathbf{v_r}||^2 = ||\mathbf{v_1}||^2 + ||\mathbf{v_2}||^2 + \ldots + ||\mathbf{v_r}||^2.$$