Linear Algebra: Exercise Sheet 7

1. Use the Gram-Schmidt process to construct orthonormal bases for $\mathbb{R}^3$ (with usual dot product) starting from the following bases:

   (a) $\{(1,0,0),(1,1,0),(1,1,1)\}$.

   (b) $\{(2,0,3),(-1,0,5),(10,-7,2)\}$.

   Write the vector $(7,4,-2)$ as a linear combination of each of the new bases.

2. Use the Gram-Schmidt process to contruct an orthonormal basis for $M(2,2)$ with inner product given by $\langle A,B\rangle = tr(B^TA)$ starting from the basis

   $$\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}.$$

   Write the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of these new basis vectors.

3. Use the Gram-Schmidt process to contruct an orthonormal basis for P_2 with inner product given by $\langle p(x), q(x)\rangle = p_0q_0 + p_1q_1 + p_2q_2$ (where $p(x) = p_0 + p_1x + p_2x^2$ and $q(x) = q_0 + q_1x + q_2x^2$) starting with the basis $\{2+3x^2, 5x^2-1, 10-7x+2x^2\}$.

4. Use the Gram-Schmidt process to construct an orthonormal basis for P_2 with inner product $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$ from the basis $\{1, x, x^2\}$. The resulting basis elements are called the first three (normalized) Legendre polynomials. Write the vector $x^2 - 2x + 3$ as a linear combination of the new basis vectors.

5. **(optional)** An $n \times n$ matrix A is called orthogonal if $A^TA = I$ (the identity matrix). Show that

   (a) A is orthogonal if and only if its columns (resp. its rows) form an orthonormal set of vectors in $\mathbb{R}^n$ (with the usual dot product).

   (b) If A is orthogonal then $(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$.

   (c) If A is orthogonal then $||A\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$.

   Interpret (b) and (c) geometrically.