Solutions to Linear Algebra coursework 1

In the following notes, we give worked solutions to each of the questions on the first coursework. Comments in italics are not part of the solutions, but advice on method and some common errors.

Question 1: Subspaces

In questions such as this one, it is often a good idea to write down the relevant definition to remind yourself what has to be checked. A **subspace** U of a vector space V over $\mathbb{F}$ is a subset U of V such that

1. $\mathbf{0} \in U$.
2. For all $\mathbf{u}, \mathbf{v} \in U$ we have $\mathbf{u} + \mathbf{v} \in U$.
3. For all $\mathbf{u} \in U$ and $\lambda \in \mathbb{F}$ we have $\lambda \mathbf{u} \in U$.

(a) Let $U = \{(x_1, x_2, \ldots, x_n) : x_1 + x_2 = 0\}$. We will check each of the subspace conditions for U in turn. *Notice the notation used for vectors and their coordinates; it is important not to get confused between the two. Also the vector is not the same as the condition $x_1 + x_2 = 0$, so writing $\mathbf{x} = x_1 + x_2 = 0$ or some such is meaningless.*
$\mathbf{0} = (0, 0, \ldots, 0)$, so the sum of the first two coordinates is $0 + 0 = 0$ and we see that $\mathbf{0} \in U$ as required.
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be in U. Then $x_1 + x_2 = y_1 + y_2 = 0$. We have $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, and the sum of the first two coordinates is $x_1 + y_1 + x_2 + y_2 = (x_1 + x_2) + (y_1 + y_2) = 0$, so $\mathbf{x} + \mathbf{y} \in U$ as required.
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in U$ (so $x_1 + x_2 = 0$). We have $\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$ and the sum of the first two coordinates is $\lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2) = 0$. Therefore $\lambda\mathbf{x} \in U$ as required, and U is a subspace of $\mathbb{R}^n$.

(b) Let $U = \{(x_1, x_2, \ldots, x_n) : x_1^2 - x_2 = 0\}$. *In order to show that a set is not a subspace, it is enough to find one of the three conditions that fails. To do this, it is in turn enough to find a single example for which the condition fails. Thus for this question, one can either try each condition in turn with a general vector* $\mathbf{x}$, *or pick an example for which it fails. Both methods are valid, but we will carry out the latter here.*
Consider $\mathbf{x} = (1, 1, 0, \ldots, 0)$ and $\mathbf{y} = (-1, 1, 0, \ldots, 0)$ in U. The sum $\mathbf{x} + \mathbf{y} = (0, 1, 0, \ldots, 0)$ has $0^2 - 1 = -1 \neq 0$ and hence does not lie in U. So U is not a subspace of $\mathbb{R}^n$. *For those who prefer not to guess examples, we could have used arbitrary* $\mathbf{x}$ *and* $\mathbf{y}$ *in U and observed that $(x_1 + y_1)^2 - (x_2 + y_2) \neq x_1^2 - x_2 + y_1^2 - y_2$ in general.*

(c) Let $U = \{(x_1, x_2, \ldots, x_n) : x_1 - x_2 + x_3 \geq 0\}$. *As in part (b), we will give an example to show the subspace conditions fail.*
Let $\mathbf{x} = (1, 0, 0, \ldots, 0)$; clearly this is a vector in U. But for $\lambda = -1 \in \mathbb{R}$ we have $\lambda\mathbf{x} = (-1, 0, 0, \ldots, 0)$ and $-1 - 0 + 0 < 0$. Thus $\lambda\mathbf{x} \notin U$, and so U is not a subspace.

(d) Let $U = \{(x_1, x_2, \ldots, x_n) : x_1 = 0 \text{ or } x_2 = 0\}$. *Many of you were confused by the word "or" here. Recall that in mathematics the phrase "A or B" means that at least one of A or B (and possibly both) is true. Those who took "or" to mean "A or B but not both" lost some (but not all!) of the available marks, as* $\mathbf{0}$ *is in U with the above definition.*
Let $\mathbf{x} = (1, 0, 0, \ldots, 0)$ and $\mathbf{y} = (0, 1, 0, \ldots, 0) \in U$. We have $\mathbf{x} + \mathbf{y} = (1, 1, 0, \ldots, 0) \notin U$, and so U is not a subspace of $\mathbb{R}^n$.

Question 2: Bases of P_2

There are several ways to approach this question, depending on whether one can remember the various results from the lectures. We will indicate a variety of methods. A **basis** for a vector space V over $\mathbb{F}$ is a set of vectors that are linearly independent and that span V.
A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is **linearly independent** if, whenever we have $\sum_{i=1}^n \lambda_i \mathbf{v}_i = 0$ with $\lambda_i \in \mathbb{F}$ for all i, then $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.
A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ **spans** V if for every vector $\mathbf{v} \in V$ we can find $\lambda_i \in \mathbb{F}$ for $1 \leq i \leq n$ such that $\mathbf{v} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$.

(a) Let $S = \{1 + x, x + x^2\}$. *Short answer:* The vector space P_2 has dimension 3, and we know that any basis of an n-dimensional space must contain exactly n elements. Thus S cannot be a basis of P_2. Therefore it cannot both be linearly independent and span. We also know that every spanning set contains a basis, and therefore S cannot span P_2 (as S does not contain 3 distinct vectors). So we only have to check whether S is linearly independent or not.
(*) Suppose $\lambda_1(1+x) + \lambda_2(x+x^2) = 0$. Then we have $\lambda_1 + (\lambda_1 + \lambda_2)x + \lambda_2 x^2 = 0 + 0x + 0x^2$. By comparing the coefficients on each side we see that $\lambda_1 = 0$ and $\lambda_2 = 0$, as required.
So S is not a basis for P_2, as it is linearly independent but does not span.
Long answer (for those who find the theorems on bases in Chapter 1 confusing!): We have to check if S is linearly independent. For this we use the argument given at (*) above. Now *(as we are not quoting any theorems)* we have to check whether S spans P_2.
Given $\mathbf{v} \in P_2$, we need to find λ_1 and $\lambda_2 \in \mathbb{R}$ such that $\mathbf{v} = \lambda_1(1+x) + \lambda_2(x+x^2)$. Any $\mathbf{v} \in P_2$ can be written in the form $\mathbf{v} = v_0 + v_1 x + v_2 x^2$ for some $v_i \in \mathbb{R}$. So we have to solve the equation

$$\lambda_1 + (\lambda_1 + \lambda_2)x + \lambda_2 x^2 = v_0 + v_1 x + v_2 x^2$$

This example is easy enough to do by hand, but we could also use Gaussian elimination. For practice, we will take the latter approach. We have to solve the system of equations

$$\begin{array}{lll} \lambda_1 & & = v_0 \\ \lambda_1 & +\lambda_2 & = v_1 \\ & \lambda_2 & = v_2 \end{array} \quad \text{or} \quad \left(\begin{array}{cc|c} 1 & 0 & v_0 \\ 1 & 1 & v_1 \\ 0 & 1 & v_2 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{cc|l} 1 & 0 & v_0 \\ 0 & 1 & v_1 - v_0 \\ 0 & 1 & v_2 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{cc|l} 1 & 0 & v_0 \\ 0 & 1 & v_1 - v_0 \\ 0 & 0 & v_2 - v_1 + v_0 \end{array}\right).$$

This is impossible to solve when $v_2 - v_1 + v_0 \neq 0$, and so the set S does not span P_2.
So S is not a basis for P_2, as it is linearly independent but does not span.

(b) Let $S = \{2x^2 - 1, 1 + 3x - 4x^2, 1 + x + x^2\}$. *Short answer:* The vector space P_2 has dimension 3, and we know that any basis of an n-dimensional space must contain exactly n elements. Thus S could be a basis of P_2. We also know that every set of n linearly independent vectors in an n-dimensional space is a basis (and similarly that every n spanning vectors form a basis), so we only need to check linear independence (or just check spanning). These are

both done in the long answer below; one only needs to be copied here...
Long answer (and completion of short answer): We first check if S is linearly independent. Suppose that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are such that

$$\lambda_1(2x^2 - 1) + \lambda_2(1 + 3x - 4x^2) + \lambda_3(1 + x + x^2) = 0 + 0x + 0x^2$$

By equating coefficients of powers of x we see that we must solve

$$\begin{array}{rrrl} -\lambda_1 & +\lambda_2 & +\lambda_3 & = 0 \\ & 3\lambda_2 & +\lambda_3 & = 0 \\ 2\lambda_1 & -4\lambda_2 & +\lambda_3 & = 0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|l} -1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -4 & 1 & 0 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|l} -1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|l} -1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 3\frac{2}{3} & 0 \end{array}\right)$$

which implies that $\lambda_3 = \lambda_2 = \lambda_1 = 0$. So S is linearly independent. *This completes the "short" answer.*
We next check if S spans P_2. Arguing as in part (a), we have to find for each triple of real numbers v_0, v_1, v_2 elements $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1(2x^2 - 1) + \lambda_2(1 + 3x - 4x^2) + \lambda_3(1 + x + x^2) = v_0 + v_1 x + v_2 x^2$$

By equating coefficients of powers of x we see that we must solve

$$\begin{array}{rrrl} -\lambda_1 & +\lambda_2 & +\lambda_3 & = v_0 \\ & 3\lambda_2 & +\lambda_3 & = v_1 \\ 2\lambda_1 & -4\lambda_2 & +\lambda_3 & = v_2 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|l} -1 & 1 & 1 & v_0 \\ 0 & 3 & 1 & v_1 \\ 2 & -4 & 1 & v_2 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|l} -1 & 1 & 1 & v_0 \\ 0 & 3 & 1 & v_1 \\ 0 & -2 & 3 & v_2 + 2v_0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|l} -1 & 1 & 1 & v_0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3}v_1 \\ 0 & 0 & 3\frac{2}{3} & v_2 + 2v_0 + \frac{2}{3}v_1 \end{array}\right).$$

This can be solved for $\lambda_1, \lambda_2, \lambda_3$, and so the set S spans P_2.
Thus S is a basis for P_2 as it is linearly independent and spans. *Notice how similar the two calculations of linear independence and spanning are; we have to carry out Gaussian elimination on two very similar augmented matrices.*

(c) Let $S = \{1 + 2x + x^2, 1 - x - 4x^2, x - x^2, 1 + 3x\}$. *By now the method of Gaussian elimination should be familiar, so we shall just give the "short" answers to the remaining two parts. For the "long" answers, check linear independence and spanning separately.* The vector space P_2 has dimension 3, and we know that any basis of an n-dimensional space must contain exactly n elements. Thus S cannot be a basis of P_2. Therefore it cannot both be linearly independent and span. We also know that it cannot be linearly independent, as

any linearly independent set has no more elements than any spanning set (and there is a set of three vectors which span P_2). So we only have to check whether S spans or not.
Arguing as in part (a), we have to find for each triple of real numbers v_0, v_1, v_2 elements $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\lambda_1(1 + 2x + x^2) + \lambda_2(1 - x - 4x^2) + \lambda_3(x - x^2) + \lambda_4(1 + 3x) = v_0 + v_1 x + v_2 x^2$$

By equating coefficients of powers of x we see that we must solve

$$\begin{array}{rrrrl} \lambda_1 & -4\lambda_2 & -\lambda_3 & & = v_0 \\ 2\lambda_1 & -\lambda_2 & +\lambda_3 & +3\lambda_4 & = v_1 \\ \lambda_1 & +\lambda_2 & & +\lambda_4 & = v_2 \end{array} \quad \text{or} \quad \left(\begin{array}{rrrr|l} 1 & -4 & -1 & 0 & v_0 \\ 2 & -1 & 1 & 3 & v_1 \\ 1 & 1 & & 1 & v_2 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrrr|l} 1 & -4 & -1 & 0 & v_0 \\ 0 & 7 & 1 & 3 & v_1 - 2v_0 \\ 0 & 5 & 1 & 0 & v_2 - v_0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrrr|l} 1 & -4 & -1 & 0 & v_0 \\ 0 & 1 & \frac{1}{7} & \frac{3}{7} & \frac{1}{7}(v_1 - 2v_0) \\ 0 & 0 & \frac{2}{7} & \frac{-15}{7} & v_2 - v_0 - \frac{5}{7}(v_1 - 2v_0) \end{array}\right).$$

This can be solved for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and so the set S spans P_2.
Thus S is not linearly independent, but it does span P_2. *Notice that the solutions to the final set of equations (for spanning) are not unique. This does not matter — for a set of vectors to span they merely have to have* some *combination equalling each vector in the space, not necessarily a* unique *combination.*

(d) Let $S = \{3 + 4x + x^2, 1 + x + x^2, 7 + 10x + x^2\}$. The vector space P_2 has dimension 3, and we know that any basis of an n-dimensional space must contain exactly n elements. Thus S could be a basis of P_2. We also know that every set of n linearly independent vectors in an n-dimensional space is a basis (and similarly that every n spanning vectors form a basis), so we only need to check linear independence (or just check spanning). If they are not linearly independent they cannot span, as such a spanning set would be a basis. Thus we check if S is linearly independent. Suppose that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are such that

$$\lambda_1(3 + 4x + x^2) + \lambda_2(1 + x + x^2) + \lambda_3(7 + 10x + x^2) = 0 + 0x + 0x^2$$

By equating coefficients of powers of x we see that we must solve

$$\begin{array}{rrrl} 3\lambda_1 & +\lambda_2 & +7\lambda_3 & = 0 \\ 4\lambda_1 & +\lambda_2 & +10\lambda_3 & = 0 \\ \lambda_1 & +\lambda_2 & +\lambda_3 & = 0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|l} 3 & 1 & 7 & 0 \\ 4 & 1 & 10 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|l} 1 & 1 & 1 & 0 \\ 3 & 1 & 7 & 0 \\ 4 & 1 & 10 & 0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|l} 1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|l} 1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which implies that we can find $\lambda_1, \lambda_2, \lambda_3 \neq 0$ satisfying our equation above (e.g. $\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 1$). So S is not linearly independent, and hence does not span either.

Question 3: Linear maps

Once again, it can be helpful to begin with the definitions to be used in answering the question.

A function $f : U \to V$ between two vector spaces over $\mathbb{F}$ is called **linear** if

1. For all $\mathbf{u}, \mathbf{v} \in U$ we have $f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$.
2. For all $\mathbf{u} \in U$ and $\lambda \in \mathbb{F}$ we have $f(\lambda\mathbf{u}) = \lambda f(\mathbf{u})$.

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (2x-y-z, x+z, y)$. We will check each of the linearity conditions in turn. *Be careful not to confuse "linear maps" with "linearly independent"! Some of you tried to show linear independence, which has nothing to do with this particular question.*

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. Then $\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$ and

$$\begin{aligned} f(\mathbf{u}+\mathbf{v}) &= (2(u_1+v_1) - (u_2+v_2) - (u_3+v_3), (u_1+v_1) + (u_3+v_3), (u_2+v_2)) \\ &= (2u_1 - u_2 - u_3, u_1 + u_3, u_2) + (2v_1 - v_2 - v_3, v_1 + v_3, v_2) \\ &= f(\mathbf{u}) + f(\mathbf{v}) \end{aligned}$$

as required. *Notice the way in which vectors are written; it is important not to omit the first and last brackets. Many of you showed that* $f(\mathbf{u}+\mathbf{v}+\mathbf{w}) = f(\mathbf{u}) + f(\mathbf{v}) + f(\mathbf{w})$ *which is more complicated (and not what the definition asks for). Also, we have not yet finished showing f is linear — some of you stopped at this point. We still need...*

Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then $\lambda\mathbf{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$ and so

$$f(\lambda\mathbf{u}) = (2\lambda u_1 - \lambda u_2 - \lambda u_3, \lambda u_1 + \lambda u_3, \lambda u_2) = \lambda(2u_1 - u_2 - u_3, u_1 + u_3, u_2) = \lambda f(\mathbf{u}).$$

We have checked that conditions (1) and (2) hold, and so f is a linear map.

(b) Let $f : M(2,2) \to \mathbb{R}^2$, with $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a - c + d + 1, \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix})$. *Again, one can show this is not linear by checking the axioms. Alternatively, if you see why it is not linear immediately, you may find it easier just to give an example. We do the latter; you may wish to try writing out the check of condition (1) as above in full instead. Some of you ignored the symbol* tr *which appears in the definition of f; this is the trace of a matrix, which you should have seen last year (and which is defined in one of my handouts).*

$f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = (0-0+0+1, 0) = (1, 0)$, and so $f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) + f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = (2, 0)$.

However, $f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right) = (0-0+0+1, 0) = (1, 0) \neq (2, 0)$.

Thus we see that f is not a linear map, as condition (1) does not hold.

(c) Let $f : P_2 \to P_2$, with $(f(p))(x) = p(x+1) + 2\frac{d}{dx}p(x)$. *This question confused many people, who either did not know how to manipulate polynomials (can you write $p(x+1)$ as $p(x) + p$?) or were confused as to what was the variable (do you check linearity by adding p and q or x and y?). In lectures, we saw a quick way to manipulate polynomials like $p(x+1)$, but this seems to have been confusing. Thus I will give a different method here which, though longer, is more likely to avoid errors.*

Any polynomial in P_2 can be written in the form $p(x) = a_0 + a_1x + a_2x^2$ with $a_0, a_1, a_2 \in \mathbb{R}$. We first need to calculate what the image of such a polynomial is under f. Now $p(x+1)$ is

just the polynomial $a_0 + a_1(x+1) + a_2(x+1)^2$, but we need to write this as sums of powers of x. We have

$$a_0 + a_1(x+1) + a_2(x+1)^2 = (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2x^2.$$

Also, the derivative of p is the polynomial $a_1 + 2a_2x$, and so we see that $f(p)$ is the polynomial of the form

$$\begin{aligned} f(p)(x) &= (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2x^2 + 2(a_1 + 2a_2x) \\ &= (a_0 + 3a_1 + a_2) + (a_1 + 6a_2)x + a_2x^2. \end{aligned}$$

Now we can check linearity. *Note: we are working with polynomials p, q etc., and not the variables x, y etc., and so we have to vary p, not x.*

Let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$. Then $(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$, and we can see that

$$\begin{aligned} f(p+q)(x) &= ((a_0+b_0) + 3(a_1+b_1) + (a_2+b_2)) + ((a_1+b_1) + 6(a_2+b_2))x + (a_2+b_2)x^2 \\ &= (a_0 + 3a_1 + a_2) + (a_1 + 6a_2)x + a_2x^2 + (b_0 + 3b_1 + b_2) + (b_1 + 6b_2)x + b_2x^2 \\ &= f(p)(x) + f(q)(x) \end{aligned}$$

as required. With $p(x)$ as above we see that $\lambda p(x) = \lambda a_0 + \lambda a_1x + \lambda a_2x^2$ and hence

$$f(\lambda p)(x) = (\lambda a_0 + 3\lambda a_1 + \lambda a_2) + (\lambda a_1 + 6\lambda a_2)x + \lambda a_2x^2 = \lambda f(p)(x)$$

as required.

We have checked that conditions (1) and (2) hold, and so f is a linear map.

A final warning: In general we have that $p(x+1) \neq p(x) + p(1)$. For example if $p(x) = x^2$ then $p(x+1) = x^2 + 2x + 1 \neq x^2 + 1$. This was a common mistake in the answers submitted.

(d) Let $f : \mathbb{C}^2 \to \mathbb{C}^2$, regarded as complex vector spaces, with $f(z_1, z_2) = (\bar{z}_2, \bar{z}_1 + z_2)$ (where $\bar{z}$ is the complex conjugate of z). *This was the question that was most likely to cause problems, and indeed it did for most people. The point is that now we must take our scalars λ to be* complex *numbers, as we are now working with* complex *vector spaces. And because of this, we have that $\bar{\lambda} \neq \lambda$ in general — which is the key to this question.*

We give an example to show f is not linear. Take $\mathbf{z} = (0, 1) \in \mathbb{C}^2$ and $\lambda = 1 + i \in \mathbb{C}$. Then

$$f(\lambda\mathbf{z}) = f(0, 1+i) = (1-i, 1+i) \neq (1+i)(1, 1) = \lambda f(\mathbf{z}).$$

So condition (2) fails to hold, and f is not a linear map.

Question 4: Images and kernels

This question is rather similar to question 2, in that there are a variety of ways to approach it, depending on which results you quote from the lectures. We will concentrate on the one which minimises the number of calculations (but will indicate other methods from time to time).

Given a linear map $f : U \to V$, the **kernel** of f is

$$\ker(f) = \{\mathbf{u} \in U : f(\mathbf{u}) = \mathbf{0}\}$$

and the **image** of f is

$$\text{im}(f) = \{\mathbf{v} \in V : f(\mathbf{u}) = \mathbf{v} \text{ for some } \mathbf{u} \in U\}.$$

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (2x - y, z, 2x + 2z - y)$. We will first determine the kernel of f.
Suppose $\mathbf{x} = (x_1, x_2, x_3) \in \ker(f)$. Then we have $(2x_1 - x_2, x_3, 2x_1 + 2x_3 - x_2) = (0, 0, 0)$. Thus we want to solve the system of equations:

$$\begin{array}{rrrl} 2x_1 & -x_2 & & = 0 \\ & & x_3 & = 0 \\ 2x_1 & -x_2 & +2x_3 & = 0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|r} 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right).$$

This is very easy to solve by inspection (or use Gaussian elimination) and we get $2x_1 = x_2$ and $x_3 = 0$. Thus a general element in the kernel of f is of the form $(x, 2x, 0)$ with $x \in \mathbb{R}$. It is easy to see that these are all vectors of the form $x(1, 2, 0)$ with $x \in \mathbb{R}$; i.e. that $(1, 2, 0)$ spans the kernel. Also any single non-zero vector forms a linearly independent set, and so $(1, 2, 0)$ is a basis of $\ker(f)$.
The rank-nullity theorem states that if $f : U \to V$ is a linear map between finite dimensional vector spaces, then

$$\dim U = \dim \ker(f) + \dim \text{im}(f).$$

Applying this to our example, where $\dim U = 3$, we see that $\dim \text{im}(f)$ must equal $3 - 1 = 2$. Thus any basis for $\text{im}(f)$ must contain precisely two elements. We also know that in an n dimensional space, any n linearly independent vectors must form a basis. So it will be enough to find 2 linearly independent vectors in $\text{im}(f)$. *If we did not want to use the rank-nullity theorem, then we would have to find such a set of independent vectors, and then also show that they span the image. We will not do this here.*
We proved in the lectures that the set of images of basis vectors under f will contain a basis for the image of f. This was not stated directly, but clearly follows from statement (*) in the proof of the rank-nullity theorem. So all we have to do is find two vectors in U whose images under f are linearly independent.
At this point there are various possibilities: (i) Pick vectors in U at random and see if their images work. (ii) Pick standard basis elements, and see if their images work. (iii) Pick vectors such that the coordinates in their images are simple to work with — e.g. ones with lots of zeros in — and see if they work. It does not matter which method you use; we will do (ii) (and as it happens our choice will be good as an example of (iii)).
We have $f(1, 0, 0) = (2, 0, 2) \in \text{im}(f)$ and $f(0, 0, 1) = (0, 1, 2) \in \text{im}(f)$. If these are linearly independent then we are done.
Suppose that $\lambda_1(2, 0, 2) + \lambda_2(0, 1, 2) = (0, 0, 0)$. Then we have $(2\lambda_1, \lambda_2, 2\lambda_1 + 2\lambda_2) = (0, 0, 0)$, and by comparing coordinates we see that $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus $(2, 0, 2)$ and $(0, 1, 2)$ are linearly independent, and hence (as the image is 2 dimensional) form a basis of $\text{im} f$.
Notice how easy this last calculation was; as we chose image vectors with zeros in different places, we did not need to bother with Gaussian elimination or similar. Don't forget that if we had not used some theorems, then we should have to check spanning also.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (x + 2y - z, z - x - 2y, 0)$. We will first determine the kernel of f.
Suppose $\mathbf{x} = (x_1, x_2, x_3) \in \ker(f)$. Then we have $(x_1 + 2x_2 - x_3, x_3 - x_1 - 2x_2, 0) = (0, 0, 0)$. Thus we want to solve the system of equations:

$$\begin{array}{rrrl} x_1 & +2x_2 & -x_3 & = 0 \\ & & 0 & = 0 \\ -x_1 & -2x_2 & +x_3 & = 0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right).$$

This is very easy to solve by inspection (we hardly even need the right-hand version, yet alone Gaussian elimination) and we get $x_3 = x_1 + 2x_2$. Thus we can choose x_1 and x_2 freely, and a general element in the kernel of f is of the form $(x, y, x + 2y)$ with $x, y \in \mathbb{R}$. Taking $x = 1$ and $y = 0$ we have $(1, 0, 1) \in \ker(f)$, and taking $x = 0$ and $y = 1$ we have $(0, 1, 2) \in \ker(f)$. We will show that these form a basis for the kernel. *We could have made other choices — there are many possible bases for the kernel. But as in part (a) this choice is simple and easy to calculate with.*
For linear independence, suppose that $\lambda_1(1, 0, 1) + \lambda_2(0, 1, 2) = (0, 0, 0)$. Then we have $(\lambda_1, \lambda_2, \lambda_1 + 2\lambda_2) = (0, 0, 0)$, and by comparing coordinates we see that $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus $(1, 0, 1)$ and $(0, 1, 2)$ are linearly independent.
For spanning, suppose that $\mathbf{u} \in \ker(f)$. By our calculations above, we know that $\mathbf{u}$ is of the form $(x, y, x + 2y)$ with $x, y \in \mathbb{R}$. But then $\mathbf{u} = x(1, 0, 1) + y(0, 1, 2)$ is a linear combination of our two vectors, and so the vectors $(1, 0, 1)$ and $(0, 1, 2)$ span the kernel of f. *Notice how our choice of vectors made the verification of spanning very easy.*
We have seen that the vectors $(1, 0, 1)$ and $(0, 1, 2)$ are linearly independent and span the kernel of f. Thus they form a basis of $\ker(f)$.
Now the rank-nullity theorem implies that $\text{im}(f)$ has dimension 1, so we just have to find some non-zero vector in $\text{im}(f)$, and that will be a basis for it. We try $f(1, 0, 0)$; this equals $(1, -1, 0)$, and so a basis for the image is given by the vector $(1, -1, 0)$.
Notice that this final vector must be non-zero — some of you had zero vectors in your supposed bases. This can never happen! If we had not used the rank-nullity theorem, we would still have to check that our image basis vector did in fact span the image. Also, rather than try to find two basis elements for the kernel first, we could have started with the image, shown it was one dimensional, and then (by the rank-nullity theorem) just found two linearly independent vectors in the kernel. Pick whichever method you find easiest.

Question 5: Vectors in coordinate form

The only serious problem that this question caused was some confusion as to which vector(s) had to be written in terms of the new basis. The question asked that each *of* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$ *be so expressed, not that the* sum $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$ *be expressed thus.*

We have to write $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in coordinates with respect to the basis $\{\mathbf{e}_1+2\mathbf{e}_2, \mathbf{e}_2+2\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3\}$. That is, we have to write each vector $\mathbf{e}_i$ in the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where $\mathbf{e}_i = a(\mathbf{e}_1+2\mathbf{e}_2)+b(\mathbf{e}_2+2\mathbf{e}_3)+c(\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3)$. First consider $\mathbf{e}_1$. We must have

$$\begin{aligned} \mathbf{e}_1 &= a(\mathbf{e}_1+2\mathbf{e}_2)+b(\mathbf{e}_2+2\mathbf{e}_3)+c(\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3) \\ &= (a+c)\mathbf{e}_1+(2a+b+c)\mathbf{e}_2+(2b-c)\mathbf{e}_3 \end{aligned}$$

and so we must solve the system of equations

$$\begin{array}{rrrl} a & & +c & =1 \\ 2a & +b & +c & =0 \\ & 2b & -c & =0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

which implies that $c=4$, $b=2$ and $a=-3$.

Similarly, for $\mathbf{e}_2$ we must solve the system of equations

$$\begin{array}{rrrl} a & & +c & =0 \\ 2a & +b & +c & =1 \\ & 2b & -c & =0 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

which implies that $c=-2$, $b=-1$ and $a=2$.

Finally, for $\mathbf{e}_3$ we must solve the system of equations

$$\begin{array}{rrrl} a & & +c & =0 \\ 2a & +b & +c & =0 \\ & 2b & -c & =1 \end{array} \quad \text{or} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right).$$

Using Gaussian elimination we get

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right) \quad \text{and then} \quad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

which implies that $c=1$, $b=1$ and $a=-1$.

Thus $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are represented by $\begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ respectively.

Question 6: The matrix of a linear map

Most attempts at this question were successful. One just has to apply the formula given in the lectures, and not confuse the various different bases. The easiest way to do this is to use the same symbols as were used in the lectures.

Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be the linear map given by $f(\mathbf{e}_1)=\mathbf{e}_2+3\mathbf{e}_3$ and $f(\mathbf{e}_2)=2\mathbf{e}_1+3\mathbf{e}_2+5\mathbf{e}_3$. We take as a basis for $U=\mathbb{R}^2$ the elements $\mathbf{u}_1=2\mathbf{e}_1+3\mathbf{e}_2$ and $\mathbf{u}_2=\mathbf{e}_1-\mathbf{e}_2$, and as a basis for $V=\mathbb{R}^3$ the elements $\mathbf{v}_1=\mathbf{e}_1$, $\mathbf{v}_2=\mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{v}_3=\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$.

The matrix representing f with respect to these bases is $A=(a_{ij})$ defined by the equations

$$f(\mathbf{u}_i)=\sum_{j=1}^{3} a_{ji}\mathbf{v}_j \quad \text{for } 1 \le i \le 2.$$

It is important to get the indices in the right order here! Now

$$\begin{aligned} f(\mathbf{u}_1) &= f(2\mathbf{e}_1+3\mathbf{e}_2)=2f(\mathbf{e}_1)+3f(\mathbf{e}_2) \\ &= 2(\mathbf{e}_2+3\mathbf{e}_3)+3(2\mathbf{e}_1+3\mathbf{e}_2+5\mathbf{e}_3) \\ &= 6\mathbf{e}_1+11\mathbf{e}_2+21\mathbf{e}_3 \\ &= a_{11}\mathbf{v}_1+a_{21}\mathbf{v}_2+a_{31}\mathbf{v}_3. \end{aligned}$$

Thus we have to solve the equation

$$6\mathbf{e}_1+11\mathbf{e}_2+21\mathbf{e}_3 = a_{11}\mathbf{e}_1+a_{21}(\mathbf{e}_1+\mathbf{e}_2)+a_{31}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3).$$

As in the preceding question, we use Gaussian elimination (or just inspect the equations) to solve this and get that $a_{31}=21$, $a_{21}=-10$, and $a_{11}=-5$.

A similar calculation for $f(\mathbf{u}_2)$ gives that $a_{12}=a_{22}=0$ and $a_{32}=-2$. Thus we deduce that

$$A=\begin{pmatrix} -5 & 0 \\ -10 & 0 \\ 21 & -2 \end{pmatrix}.$$

As long as the various $\mathbf{u}$*'s and* $\mathbf{v}$*'s are not confused, this is one of the easier questions to complete — and most of you solved it correctly. The most common mistake was to write down the transpose of A rather than A itself.*