Linear Algebra: Solutions to Coursework 1

1. (a) $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_4 = 0\}$ is a subspace of $\mathbb{R}^4$. In order to prove it, we need to check conditions (S1),(S2) and (S3).
(S1) $(0,0,0,0) \in U$, as $x_1 + x_4 = 0 + 0 = 0$.
(S2) If $(x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4) \in U$, i.e. $x_1 + x_4 = 0$ and $x'_1 + x'_4 = 0$, then

$$(x_1, x_2, x_3, x_4) + (x'_1, x'_2, x'_3, x'_4) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3, x_4 + x'_4) \in U$$

as

$$(x_1 + x'_1) + (x_4 + x'_4) = (x_1 + x_4) + (x'_1 + x'_4) = 0 + 0 = 0.$$

(S3) If $(x_1, x_2, x_3, x_4) \in U$ and $\lambda \in \mathbb{R}$ then

$$\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) \in U$$

as $(\lambda x_1) + (\lambda x_4) = \lambda(x_1 + x_4) = \lambda 0 = 0$.

(b) $U = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d\}$ is a subspace of $M(2,2)$. In order to prove it, we need to check conditions (S1),(S2) and (S3).
(S1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ as $0 + 0 = 0 + 0$.
(S2) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix} \in U$$

as $(a + a') + (b + b') = (a + b) + (a' + b') = (c + d) + (c' + d') = (c + c') + (d + d')$.
(S3) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ and $\lambda \in \mathbb{R}$ then

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \in U$$

as $(\lambda a) + (\lambda b) = \lambda(a + b) = \lambda(c + d) = (\lambda c) + (\lambda d)$.

(c) $U = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = 1\}$ is not a subspace of $\mathbb{R}^{\mathbb{R}}$. It is enough to show that one condition fails. Take for example (S1). The zero vector is the zero function $\theta : \mathbb{R} \to \mathbb{R}$ defined by $\theta(x) = 0$ for all $x \in \mathbb{R}$. In particular, $\theta(0) = 0 \neq 1$, thus $\theta \notin U$.

2. (a) $U \cap W$ is always a subspace of V. In order to prove it we need to show that (S1),(S2) and (S3) are satisfied.
(S1) As the zero vector $\mathbf{0} \in U$ and $\mathbf{0} \in W$, we have $\mathbf{0} \in U \cap W$.
(S2) Take $\mathbf{v}, \mathbf{u} \in U \cap W$. In particular, $\mathbf{v}, \mathbf{u} \in U$ so (as U is a subspace) we have $\mathbf{v} + \mathbf{u} \in U$. But also $\mathbf{v}, \mathbf{u} \in W$ so (as W is a subspace) we have $\mathbf{v} + \mathbf{u} \in W$. Thus we have $\mathbf{v} + \mathbf{u} \in U \cap W$.
(S3) Take $\mathbf{v} \in U \cap W$ and $\lambda \in \mathbb{R}$. In particular, $\mathbf{v} \in U$, so (as U is a subspace) we have $\lambda\mathbf{v} \in U$. But also $\mathbf{v} \in W$, so (as W is a subspace) we have $\lambda\mathbf{v} \in W$. Thus we have $\lambda\mathbf{v} \in U \cap W$.

(b) $U \cup W$ is not always a subspace of V. Take for example $V = \mathbf{R}^2$, $U = \{(x,0) \mid x \in \mathbb{R}\}$ and $W = \{(0,y) \mid y \in \mathbb{R}\}$. Then it is easy to check that U and W are subspaces of V. However, $U \cup W = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \,\text{or}\, y = 0\}$ is not a subspace of V. How to see that? Show for example that (S2) fails. Take $(1,0) \in U \cup W$ and $(0,1) \in U \cup W$, then $(1,0) + (0,1) = (1,1) \notin U \cup W$.

3. (a) As this set contains 4 vectors and $\dim \mathbb{R}^3 = 3$ this set cannot be linearly independent (using Corollary 1.24 from the lecture), in particular, it is not a basis. Is it spanning? To check this, we need to check whether we can write any vector $(x,y,z) \in \mathbb{R}^3$ as a linear combination of these vectors, i.e. we want to find some $\lambda_1, \lambda_2, \lambda_3$ and λ_4 (depending on x, y, z) satisfying

$$(x,y,z) = \lambda_1(0,0,1) + \lambda_2(1,0,1) + \lambda_3(0,1,0) + \lambda_4(-1,-1,0).$$

This is equivalent to the following system of linear equations

$$\begin{cases} x &= \lambda_2 - \lambda_4 \\ y &= \lambda_3 - \lambda_4 \\ z &= \lambda_1 + \lambda_2 \end{cases}$$

Solving this system we get $\lambda_1 = z - x - \lambda_4$, $\lambda_2 = x + \lambda_4$, $\lambda_3 = y + \lambda_4$ and λ_4 can be any real number. So we could take for example $\lambda_4 = 0$, $\lambda_1 = z - x$, $\lambda_2 = x$ and $\lambda_3 = y$. Thus this set is a spanning set for $\mathbb{R}^3$.

(b) As this set contains 3 vectors and $\dim \mathbb{R}^4 = 4$ this set cannot be spanning (using Corollary 1.24 from the lecture), in particular, it is not a basis. Is it linearly independent? Write

$$\lambda_1(0,0,0,1) + \lambda_2(3,0,1,0) + \lambda_3(5,4,3,-2) = (0,0,0,0).$$

Does this equation implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$? Rewrite this equation as

$$\begin{cases} 0 &= 3\lambda_2 + 5\lambda_3 \\ 0 &= 4\lambda_3 \\ 0 &= \lambda_2 + 3\lambda_3 \\ 0 &= \lambda_1 - 2\lambda_3 \end{cases}$$

The only solution to this system of linear equation is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus this set is linearly independent.

(c) Is this set linearly independent?

$$\lambda_1(3) + \lambda_2(2 - x) + \lambda_3(4 + x - x^2) = 0$$

Does this implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$? We have

$$(3\lambda_1 + 2\lambda_2 + 4\lambda_3) + (-\lambda_2 + \lambda_3)x + (-\lambda_3)x^2 = 0$$

so

$$\begin{cases} 0 &= 3\lambda_1 + 2\lambda_2 + 4\lambda_3 \\ 0 &= -\lambda_2 + \lambda_3 \\ 0 &= -\lambda_3 \end{cases}$$

But the only solution to this system is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus this set is linearly independent.

Now, as $\dim P_2 = 3$ and we have a set containing 3 linearly independent vectors, it is automatically a basis (and hence also a spanning set), using Corollary 1.24 from the lecture.

4. (a) This map is linear. In order to prove it we need to check the following two conditions.
(i) $f(p(x)+q(x)) = f(p(x)) + f(q(x))$ for all $p(x), q(x) \in P_n$.
Now

$$\begin{aligned} f(p(x)+q(x)) &= x^2\frac{d}{dx}(p(x)+q(x)) \\ &= x^2(\frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))) \\ &= x^2\frac{d}{dx}(p(x)) + x^2\frac{d}{dx}(q(x)) \\ &= f(p(x)) + f(q(x)). \end{aligned}$$

(ii) $f(\lambda p(x)) = \lambda f(p(x))$ for all $p(x) \in P_n$ and all $\lambda \in \mathbb{R}$.
Now

$$f(\lambda p(x)) = x^2\frac{d}{dx}(\lambda p(x)) = x^2\lambda\frac{d}{dx}(p(x)) = \lambda(x^2\frac{d}{dx}(p(x))) = \lambda f(p(x)).$$

(b) This map is not linear. In order to prove it is it enough to show that one of the two conditions fails. Take for example (i). Pick $(0,0,1)$ and $(1,1,-1)$ then

$$f((0,0,1)+(1,1,-1)) = f(1,1,0) = (1+1)0 = 0$$

but

$$f(0,0,1) + f(1,1,-1) = (0+0)1 + (1+1)(-1) = -2$$

showing that (i) fails.

5. (a) No there isn't such a linear map as if there were one then we would have

$$f(4,2) = f(2(1,1)+(2,0)) = 2f(1,1)+f(2,0) = 2(1,0,0)+(1,2,3) = (3,2,3) \neq (0,0,-5).$$

(b) Using the calculation above we see that there is such a linear map. To find $f(x,y)$, first write

$$(x,y) = a(1,1) + b(2,0) = (a+2b, a)$$

Thus we must have $a = y$ and $b = \frac{1}{2}(x-y)$. Now

$$\begin{aligned} f(x,y) &= f(y(1,1) + \frac{1}{2}(x-y)(2,0)) \\ &= yf(1,1) + \frac{1}{2}(x-y)f(2,0) \\ &= y(1,0,0) + \frac{1}{2}(x-y)(1,2,3) \\ &= (\frac{1}{2}(x+y), x-y, \frac{3}{2}(x-y)). \end{aligned}$$