Solutions to Linear Algebra coursework 2

In the following notes, we give worked solutions to each of the questions on the second coursework. Comments in italics are not part of the solutions, but advice on method and some common errors.

Most students find the second half of the course rather easier than the first half, as the questions (being about actual matrices rather than general vectors) are more computational and less abstract. However, we do need some of the basic ideas from the first half (e.g. linear independence, bases...) to do some of the questions.

Question 1: Eigenvalues and eigenvectors I

Eigenvalues of a matrix A are solutions of the equation $\det(A-\lambda I)=0$. *Note that this is not the definition of an eigenvalue, but it is the easiest way to calculate them for a given matrix. If you are asked to define the eigenvalues of a matrix, you should use Definition 3.1 from the course notes (and hence first define eigenvectors).*

We have

$$\det(A-\lambda I)=\begin{vmatrix} -17-\lambda & 0 & 8 \\ -6 & 1-\lambda & 2 \\ -48 & 0 & 23-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)(\lambda-7)$$

and hence the eigenvalues of A are $\lambda=\pm1$ and $\lambda=7$. *To save space I have omitted the intermediate steps of this (and later) calculations. In an exam you should include details of all calculations — if you make a mistake halfway the examiner can then give you some credit for the part which is correct!*

First consider the case $\lambda=-1$. Recall that $\mathbf{x}$ is an **eigenvector** with eigenvalue λ is $\mathbf{x}$ is a solution of the equation $A\mathbf{x}=\lambda\mathbf{x}$. Thus we need to solve the equation $A\mathbf{x}=-\mathbf{x}$; i.e.

$$\begin{array}{llll} -17x_1 & & +8x_3 & =-x_1 \\ -6x_2 & +x_2 & +2x_3 & =-x_2 \\ -48x_1 & & +23x_3 & =-x_3 \end{array} \quad \text{or} \quad \begin{array}{llll} -16x_1 & & +8x_3 & =0 \\ -6x_2 & +2x_2 & +2x_3 & =0 \\ -48x_1 & & +24x_3 & =0. \end{array}$$

Using Gaussian elimination (*which by now you should be happy with; certainly most of you had no problems here. For this reason I shall omit some of the intermediate steps... Remember though that you are not allowed to use* column *operations, only rows!*) we see that

$$\left(\begin{array}{ccc|c} -16 & 0 & 8 & 0 \\ -6 & 2 & 2 & 0 \\ -48 & 0 & 24 & 0 \end{array}\right) \quad \text{reduces to} \quad \left(\begin{array}{ccc|c} -16 & 0 & 8 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and hence that the general solution is $2x_1=x_3$ and $2x_2=x_3$. Thus the eigenspace $S_A(-1)$ consists of all vectors of the form $\begin{pmatrix} x \\ x \\ 2x \end{pmatrix}$; all such vectors are scalar multiples of $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ which thus forms a basis for $S_A(-1)$. *Be careful not to get confused when converting the general solution into a vector. The equations $2x_1=x_3$ and $2x_2=x_3$ correspond to solutions* $\begin{pmatrix} x \\ x \\ 2x \end{pmatrix}$, *not* $\begin{pmatrix} 2x \\ 2x \\ x \end{pmatrix}$. *This was quite a common mistake.*

Next consider the case $\lambda=1$. We need to solve the equation $A\mathbf{x}=\mathbf{x}$; i.e.

$$\begin{array}{llll} -17x_1 & & +8x_3 & =x_1 \\ -6x_1 & +x_2 & +2x_3 & =x_2 \\ -48x_1 & & +23x_3 & =x_3 \end{array} \quad \text{or} \quad \begin{array}{lll} -18x_1 & +8x_3 & =0 \\ -6x_1 & +2x_3 & =0 \\ -48x_1 & +22x_3 & =0. \end{array}$$

Arguing in the same way as for the case $\lambda=-1$ you should find that the general solution to this is $x_1=x_3=0$ and x_2 is anything. Thus the eigenspace $S_A(1)$ consists of all vectors of the form $\begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$; all such vectors are scalar multiples of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ which thus forms a basis for $S_A(1)$.

Finally consider the case $\lambda=7$. We need to solve the equation $A\mathbf{x}=7\mathbf{x}$; i.e.

$$\begin{array}{llll} -17x_1 & & +8x_3 & =7x_1 \\ -6x_1 & +x_2 & +2x_3 & =7x_2 \\ -48x_1 & & +23x_3 & =7x_3 \end{array} \quad \text{or} \quad \begin{array}{llll} -24x_1 & & +8x_3 & =0 \\ -6x_1 & -6x_2 & +2x_3 & =0 \\ -48x_1 & & +16x_3 & =0. \end{array}$$

Arguing in the same way as for the case $\lambda=1$ you should find that the general solution to this is $x_3=3x_1$ and $x_2=0$. Thus the eigenspace $S_A(7)$ consists of all vectors of the form $\begin{pmatrix} x \\ 0 \\ 3x \end{pmatrix}$; all such vectors are scalar multiples of $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ which thus forms a basis for $S_A(7)$.

Notice that in each case above the eigenspace was 1-dimensional. Thus we did not have to check linear independence of basis elements, and it was easy to see that they spanned. This will not always be the case, as we shall see in the next question. Also note that a basis vector can **never** *(by definition) equal the zero vector!*

Finally, if λ is an eigenvalue then there must be some **non-zero** *eigenvector with that eigenvalue. If you cannot find one then you must have made a mistake (either in your calculation of eigenvalues or of eigenvectors).*

Question 2: Eigenvalues and eigenvectors II

As in Question 1, we first need to find all solutions of the equation $\det(A-\lambda I)=0$. We have

$$\det(A-\lambda I)=\begin{vmatrix} 5-\lambda & -2 & -2 \\ 0 & 3-\lambda & 0 \\ 4 & -4 & -1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-3)^2$$

and hence the eigenvalues of A are $\lambda=1$ and $\lambda=3$.

First consider the case $\lambda = 1$. We need to solve the equation $A\mathbf{x} = \mathbf{x}$; i.e.

$$\begin{array}{rrrl} 5x_1 & -2x_2 & -2x_3 & = x_1 \\ & 3x_2 & & = x_2 \\ 4x_1 & -4x_2 & -x_3 & = x_3 \end{array} \qquad \text{or} \qquad \begin{array}{rrrl} 4x_1 & -2x_2 & -2x_3 & = 0 \\ & 2x_2 & & = 0 \\ 4x_1 & -4x_2 & -2x_3 & = 0. \end{array}$$

Using Gaussian elimination (or otherwise — you should make sure you can fill in the missing steps as in Question 1) we see that the general solution is $2x_1 = x_3$ and $x_2 = 0$. Thus $S_A(1)$ consists of all vectors of the form $\begin{pmatrix} x \\ 0 \\ 2x \end{pmatrix}$; all such are scalar multiples of $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ which thus forms a basis for $S_A(1)$.

Next consider the case $\lambda = 3$. We need to solve the equation $A\mathbf{x} = 3\mathbf{x}$; i.e.

$$\begin{array}{rrrl} 5x_1 & -2x_2 & -2x_3 & = 3x_1 \\ & 3x_2 & & = 3x_2 \\ 4x_1 & -4x_2 & -x_3 & = 3x_3 \end{array} \qquad \text{or} \qquad \begin{array}{rrrl} 2x_1 & -2x_2 & -2x_3 & = 0 \\ & 0 & & = 0 \\ 4x_1 & -4x_2 & -4x_3 & = 0. \end{array}$$

Using Gaussian elimination (or otherwise — again you should make sure you can fill in the missing steps) we see that the general solution is $x_1 - x_2 - x_3 = 0$. As we have one equation in three unknowns, we can choose two of them freely (and the third will then be fixed). If we let $x_1 = a$ and $x_2 = b$ then $S_A(3)$ consists of all vectors of the form $\begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$. *We could have made other choices, for example fixing x_2 and x_3, which might lead to different eigenvectors in the end. But (as long as we did not make a mistake!) these would still give a valid answer.*

We next need to find a basis of $S_A(3)$. Taking $a = 1$ and $b = 0$ (respectively $a = 0$ and $b = 1$) we see that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are two vectors in $S_A(3)$. We must show they form a basis.

There are several important things to note here. First, we could have picked other possible pairs of vectors. These would also work, as long as we were not unlucky enough to pick two which were linearly dependent! To avoid this problem (and make the rest of the calculations easy) we have picked two simple vectors (with small numbers) containing zeros in different positions. We could however have chosen differently.

Second, we cannot stop here, but must check that we have actually got a basis of $S_A(3)$! It is possible that we might have linearly dependent vectors, or that $S_A(3)$ has more than two vectors in a basis. The latter cannot occur as $S_A(3)$ is a subspace of $\mathbb{R}^3$, and does not contain $S_A(1)$, so must have dimension less than $3 = \dim \mathbb{R}^3$ by a theorem in the course. But we do not need to say this — it is enough to find a set of vectors which are linearly independent and span. As a general rule, if you can write an arbitrary vector using k free variables and all the rest as functions of them (as here, where a and b are chosen freely, while $a - b$ depends on them) then the vector space has dimension k. You may wish to remember this as a way

to check you have a basis of the right size, but it does not allow you to avoid checking that your set is actually a basis.

To show that our two vectors form a basis of $S_A(3)$, we must show that they are linearly independent and span. For linear independence, suppose that

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we must have $\lambda_1 = \lambda_2 = 0$, and hence the two vectors are linearly independent. To see that they span, we must show that a general vector in $S_A(3)$ can be written as a linear combination of them. Thus we must show that we can find λ_1 and λ_2 such that

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$$

for any pair of scalars a and b. But clearly $\lambda_1 = a$ and $\lambda_2 = b$ satisfy this. We have therefore shown that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ form a basis of $S_A(3)$.

To finish the question we must find an invertible matrix P such that the product $P^{-1}AP$ is diagonal. We know by Theorem 3.8 (the Diagonalisation Theorem) how to do this, but I think that you should either state this Theorem or just write down P, calculate its inverse, and check that the product is diagonal. If you do neither, an examiner will not know how you arrived at P, and hence if you have made a mistake will not be able to give you any marks for it.

Vectors from distinct eigenspaces are linearly independent. Hence by the Diagonalisation Theorem (as we have found three linearly independent eigenvectors in a three-dimensional space) we may form the matrix P whose columns are these eigenvectors and this will be invertible with $P^{-1}AP$ diagonal. Indeed, the diagonal entries in the product will be the eigenvalues of the corresponding eigenvectors. Thus we may take $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$. *Note that we could have chosen a different P by reordering the columns.* By direct calculation (*Check!*) we see that $P^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and (*without having to calculate, since we have quoted the Diagonalisation Theorem*) that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ ($= D$ say).

It remains to calculate A^{10}. As $A = PDP^{-1}$ we have

$$A^{10} = (PDP^{-1})^{10} = PDP^{-1}PD \dots P^{-1}PDP^{-1} = PD^{10}P^{-1}$$

and hence

$$\begin{aligned} A^{10} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{10} \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 59049 & 0 \\ 0 & 0 & 59049 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 118097 & -59048 & -59048 \\ 0 & 59049 & 0 \\ 118096 & -118096 & -59047 \end{pmatrix}. \end{aligned}$$

Question 3: Inner products

As usual, it can be helpful to begin with the definitions to be used in answering the question. A **real inner product** on a real vector space V is a function that assigns to each pair of vectors $\mathbf{u}, \mathbf{v}$ in V a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ such that

1. For all $\mathbf{u}, \mathbf{v} \in V$ we have $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ we have $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
3. For all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{R}$ we have $\langle \lambda\mathbf{u}, \mathbf{v} \rangle = \lambda\langle \mathbf{u}, \mathbf{v} \rangle$.
4. For all $\mathbf{u} \in V$ we have $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$.

Of the four properties above, the fourth is the most unlikely to hold in any given example. Thus you may wish to start by checking this one, as if it fails you need not check any more. In the following solutions we will just show that one of the four fails (if any do) — there may be others that fail, and you could check those instead. Also, remember that you only need to find one example of vectors that fail to work; you may find this easier than working with an abstract vector. Either way is fine though, as long as you get the right answer.

(a) Let $\langle -, - \rangle$ on $\mathbb{R}^3$ be given by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_3y_3$. We will start (*as suggested above*) by checking the fourth condition. Let $\mathbf{u} = (0,1,0) \in \mathbb{R}^3$. Then $\langle \mathbf{u}, \mathbf{u} \rangle = 0^2 + 0^2 = 0$ and so condition (4) fails. Thus this is not an inner product.

(b) Let $\langle -, - \rangle$ on $\mathbb{R}^3$ be given by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$. *We could again start by checking (4), but for once this condition holds (you may wish to check this). So to save space we will jump straight to checking condition (3) (we could also have checked (2) which it also fails).*
Let $\mathbf{u} = (0,1,0) \in \mathbb{R}^3$ and $\lambda = 2$. Then $\lambda\langle \mathbf{u}, \mathbf{u} \rangle = 2(0^2 + 1^2 + 0^2) = 2$, but $\langle \lambda\mathbf{u}, \mathbf{u} \rangle = (0^2 + 2^2 \times 1^2 + 0^2) = 4$. Therefore $\lambda\langle \mathbf{u}, \mathbf{u} \rangle \neq \langle \lambda\mathbf{u}, \mathbf{u} \rangle$ and so condition (3) fails. Thus this is not an inner product.

(c) Let $\langle -, - \rangle$ on $M(2,2)$ be given by $\langle A, B \rangle = \det(BA)$. *We check condition (4) first.* Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\langle A, A \rangle = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and so condition (4) fails. Thus this is not an inner product. (*It also fails (2).*)

(d) Let $\langle -, - \rangle$ on $M(2,2)$ be given by $\langle A, B \rangle = \operatorname{tr}(AB)$. *Again we check condition (4) first.* Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\langle A, A \rangle = \operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and so condition (4) fails. Thus this is not an inner product.

(e) Let $\langle -, - \rangle$ on P_2 be given by $\langle p, q \rangle = p(0)q(0) + p(\frac{1}{2})q(\frac{1}{2}) + p(1)q(1)$. We check each condition in turn. Condition (1): Let $p, q \in P_2$. We have

$$\begin{aligned} \langle p, q \rangle &= p(0)q(0) + p(\tfrac{1}{2})q(\tfrac{1}{2}) + p(1)q(1) \\ &= q(0)p(0) + q(\tfrac{1}{2})p(\tfrac{1}{2}) + q(1)p(1) = \langle q, p \rangle. \end{aligned}$$

Condition (2): Let $p, q, r \in P_2$. We have

$$\begin{aligned} \langle p+q, r \rangle &= (p(0)+q(0))(r(0)) + (p(\tfrac{1}{2})+q(\tfrac{1}{2}))(r(\tfrac{1}{2})) + (p(1)+q(1))(r(1)) \\ &= p(0)r(0) + p(\tfrac{1}{2})r(\tfrac{1}{2}) + p(1)r(1) + q(0)r(0) + q(\tfrac{1}{2})r(\tfrac{1}{2}) + q(1)r(1) \\ &= \langle p, r \rangle + \langle q, r \rangle. \end{aligned}$$

Condition (3): Let $p, q \in P_2$ and $\lambda \in \mathbb{R}$. We have

$$\begin{aligned} \langle \lambda p, q \rangle &= \lambda p(0)q(0) + \lambda p(\tfrac{1}{2})q(\tfrac{1}{2}) + \lambda p(1)q(1) \\ &= \lambda(p(0)q(0) + p(\tfrac{1}{2})q(\tfrac{1}{2}) + p(1)q(1)) = \lambda\langle p, q \rangle. \end{aligned}$$

Condition (3): Let $p \in P_2$. We have

$$\begin{aligned} \langle p, p \rangle &= p(0)p(0) + p(\tfrac{1}{2})p(\tfrac{1}{2}) + p(1)p(1) \\ &= p(0)^2 + p(\tfrac{1}{2})^2 + p(1)^2 \geq 0 \end{aligned}$$

Further, $\langle p, p \rangle = 0$ if and only if $p(0) = p(\frac{1}{2}) = p(1) = 0$. But any non-zero polynomial of degree at most 2 has at most 2 distinct roots. Therefore $\langle p, p \rangle = 0$ if and only if $p = 0$.
We have checked that conditions (1)-(4) hold, so this is an inner product.

Note the last paragraph of part (e). You have to explain why p is zero given the fact that $p(0) = p(\frac{1}{2}) = p(1) = 0$. If we had worked in P_3 this would **not** *have followed, as there the polynomial $p(x) = x(x - \frac{1}{2})(x - 1)$ would have been a counterexample.*

Question 4: Orthonormal bases

As usual, it may help to recall the definitions needed for the question. Given an inner product $\langle -, - \rangle$ on a real vector space V, a set S of vectors is called **orthogonal** if every pair $\mathbf{u}, \mathbf{v}$ of distinct vectors from S satisfies $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. A set S of vectors is called **orthonormal** if it is an orthogonal set, and ever vector $\mathbf{u}$ satisfies $\langle \mathbf{u}, \mathbf{u} \rangle = 1$. The **dot product** on $\mathbb{R}^4$ is given by $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{4} u_iv_i$.

To show that the given set (which to save space I will refer to as $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$) is orthogonal we must check that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_4 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_4 \rangle = \langle \mathbf{u}_3, \mathbf{u}_4 \rangle = 0.$$

Note we do not need to check the other distinct pairs, e.g. $\langle \mathbf{u}_2, \mathbf{u}_1 \rangle = 0$ as we know that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ by the properties of an inner product. To show that S is an orthonormal set we also need to check that

$$\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = \langle \mathbf{u}_4, \mathbf{u}_4 \rangle = 1.$$

Both these checks are easy, and were carried out successfully by everyone who attempted them, so I omit them here. **However** *we also need to show that S is a* **basis** *of $\mathbb{R}^4$. When marking the coursework I did not deduct marks for this — but you should have included some reason, e.g.:*

Any set of orthogonal vectors are linearly independent (*this was Theorem 4.12 in the course*), and so S is a set of 4 linearly independent vectors in a 4-dimensional space. Therefore (*by Corollary 1.32 in the course*) S forms a basis of $\mathbb{R}^4$.

To complete the question you have to write some vector as a linear combination of these basis elements. There are two ways to do this: (i) try to solve 4 equations in 4 unknowns (which is very long and tedious, especially as the final answers are not particularly nice fractions!) or (ii) Use a Theorem from the course to make life much easier. Naturally, we will do the latter.

We know that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthonormal basis for a vector space V, then any $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2\rangle \mathbf{u}_2 + \langle \mathbf{v}, \mathbf{u}_3\rangle \mathbf{u}_3 + \langle \mathbf{v}, \mathbf{u}_4\rangle \mathbf{u}_4.$$

You should check that in the example in the question we get

$$\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{u}_1 + \frac{2}{\sqrt{15}}\mathbf{u}_2 + \frac{9}{\sqrt{15}}\mathbf{u}_3 - \frac{12}{\sqrt{3}}\mathbf{u}_4.$$

Note that you can check this answer to see if you have made a mistake, by adding the four vectors. If you have time in an exam this is always worth doing.

Question 5: Gram-Schmidt

Provided that the algorithm is written out correctly (and no computational mistakes occur) this is quite a straightforward question, as it becomes one long calculation. However it is very easy to make a mistake, so checking the answer at various stages can be useful.

Given a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of a real vector space V, the **Gram-Schmidt process** constructs an orthogonal basis for V in the following manner. Proceeding by induction on i we define new vectors $\mathbf{w}_i$ and $\mathbf{v}_i$ by setting

$$\mathbf{w}_i = \mathbf{u}_i - \sum_{j=1}^{i-1} \langle \mathbf{u}_i, \mathbf{v}_j\rangle \mathbf{v}_j \quad \text{and} \quad \mathbf{v}_i = \frac{\mathbf{w}_i}{\| \mathbf{w}_i \|} \tag{1}$$

where $\| \mathbf{u} \| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for V.

We will apply this to $M(2,2)$ with inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$ and basis

$$\left\{\mathbf{u}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

To save space, we will omit the calculations of $\mathrm{tr}(B^T A)$; you should check that you can obtain the answers given at each stage.

First we apply (1) with $i = 1$. Here $\mathbf{w}_1 = \mathbf{u}_1$ (as the sum is empty) and $\langle \mathbf{u}_1, \mathbf{u}_1\rangle = 2$, so $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Next we apply (1) with $i = 2$. Now $\mathbf{w}_2 = \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1\rangle \mathbf{v}_1$ so

$$\mathbf{w}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - \langle \mathbf{u}_2, \mathbf{v}_1\rangle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

and $\mathbf{v}_2 = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$.

Continuing in this way you should check that

$$\mathbf{w}_3 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{pmatrix} \quad \mathbf{v}_3 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \quad \mathbf{w}_4 = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and so the set

$$\left\{\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, \mathbf{v}_3 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right\}$$

is an orthonormal basis of $M(2,2)$.

If you have time in an exam you may want to check that these vectors are indeed orthogonal and orthonormal.

Question 6: Orthogonal diagonalisation

This question combines just about all the topics introduced during the course. We start by calculating eigenvalues and eigenspaces, then find bases for these spaces, then use Gram-Schmidt to make these bases orthonormal, and then use the resulting vectors to diagonalise the given matrix. Because of this range of topics, such a question can be popular with examiners.

We first have to calculate the eigenvalues of A; i.e. solutions of the equation $\det(A - \lambda I) = 0$. We have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & \frac{1}{2}-\lambda & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{1}{2}-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-2)^2$$

and hence the eigenvalues of A are $\lambda = 2$ and $\lambda = -1$.

First consider the case $\lambda = -1$. Recall that $\mathbf{x}$ is an **eigenvector** with eigenvalue λ is $\mathbf{x}$ is a solution of the equation $A\mathbf{x} = \lambda\mathbf{x}$. Thus we need to solve the equation $A\mathbf{x} = -\mathbf{x}$; i.e.

$$\begin{aligned} 2x_1 \qquad\qquad &= -x_1 \\ \tfrac{1}{2}x_2 + \tfrac{3}{2}x_3 &= -x_2 \\ \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 &= -x_3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3x_1 \qquad\qquad &= 0 \\ \tfrac{3}{2}x_2 + \tfrac{3}{2}x_3 &= 0 \\ \tfrac{3}{2}x_2 + \tfrac{3}{2}x_3 &= 0. \end{aligned}$$

Clearly in this case the general solution is $x_1 = 0$ and $x_2 = -x_3$. Thus the eigenspace $S_A(-1)$ consists of all vectors of the form $\begin{pmatrix} 0 \\ x \\ -x \end{pmatrix}$; all such vectors are scalar multiples of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ which thus forms a basis for $S_A(-1)$.

Next consider the case $\lambda = 2$. We need to solve the equation $A\mathbf{x} = 2\mathbf{x}$; i.e.

$$\begin{array}{llll} 2x_1 & & & = 2x_1 \\ & \frac{1}{2}x_2 & +\frac{3}{2}x_3 & = 2x_2 \\ & \frac{3}{2}x_2 & +\frac{1}{2}x_3 & = 2x_3 \end{array} \quad \text{or} \quad \begin{array}{llll} 0 & & & = 0 \\ & -\frac{3}{2}x_2 & +\frac{3}{2}x_3 & = 0 \\ & \frac{3}{2}x_2 & -\frac{3}{2}x_3 & = 0. \end{array}$$

Clearly in this case the general solution is $x_2 = x_3$ with x_1 chosen freely. If we let $x_1 = a$ and $x_2 = b$ then $S_A(2)$ consists of all vectors of the form $\begin{pmatrix} a \\ b \\ b \end{pmatrix}$. *We could have made other choices, for example fixing x_2 and x_3, which might lead to different eigenvectors in the end. But (as long as we did not make a mistake!) these would still give a valid answer.*
We next need to find a basis of $S_A(2)$. Taking $a = 1$ and $b = 0$ (respectively $a = 0$ and $b = 1$) we see that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are two vectors in $S_A(2)$. We must show they form a basis. *As in Question 2, we have chosen simple values of a and b, and for the same reasons as there.*

To show that our two vectors form a basis of $S_A(2)$, we must show that they are linearly independent and span. For linear independence, suppose that

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we must have $\lambda_1 = \lambda_2 = 0$, and hence the two vectors are linearly independent. To see that they span, we must show that a general vector in $S_A(2)$ can be written as a linear combination of them. Thus we must show that we can find λ_1 and λ_2 such that

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ b \end{pmatrix}$$

for any pair of scalars a and b. But clearly $\lambda_1 = a$ and $\lambda_2 = b$ satisfy this. We have therefore shown that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis of $S_A(2)$.

To finish the question we must find an **orthogonal** *matrix P such that the product $P^T AP$ is diagonal. We know by Theorem 3.8 (the Diagonalisation Theorem) how to find some invertible matrix P, but in general this will not be orthogonal. By Theorem 4.20 (the Orthogonal Diagonalisation Theorem) we need to find an* **orthonormal** *basis of eigenvectors. Thus we need to convert our basis elements above using the Gram-Schmidt process.*

As we want an orthogonal matrix at the end, we will need an orthonormal basis of $\mathbb{R}^3$. As A is symmetric we know (*by Theorem 4.21*) that eigenvectors from different eigenspaces will be orthogonal. Thus we can apply the Gram-Schmidt process to each eigenspace separately.

First consider $S_A(-1)$. Our basis vector here was $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Applying Gram-Schmidt we just set $\mathbf{v} = (\| \mathbf{u} \|)^{-1}\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Next consider $S_A(2)$. Our basis vectors here were $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Applying Gram-Schmidt we get $\mathbf{v}_1 = (\| \mathbf{u}_1 \|)^{-1}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and find (*Check!*) that $\mathbf{w}_2 = \mathbf{u}_2$.

Finally, $\mathbf{v}_2 = (\| \mathbf{w}_2 \|)^{-1}\mathbf{w}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. *We have here saved space by not explaining the Gram-Schmidt process; really we should have described what we were doing (and said how we chose each vector) as in Question 5. Notice too that we applied Gram-Schmidt separately to each eigenspace, as noted above. You* can *apply it to all three vectors in one go, but this is more complicated (and unnecessary).*

We have found a set of three orthonormal eigenvectors for A. Hence by the Orthogonal Diagonalisation Theorem we may form the matrix P whose columns are these eigenvectors and this will be orthogonal with $P^T AP$ diagonal. Indeed, the diagonal entries in the product will be the eigenvalues of the corresponding eigenvectors. Thus we may take $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$.

Then $P^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and (*without having to calculate, since we have quoted the Orthogonal Diagonalisation Theorem*) that $P^T AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.