Linear Algebra: Solutions to Coursework 2

1.

$$\begin{aligned}\operatorname{Ker} f &= \{(x,y,z,t)\in\mathbb{R}^4 \mid x+y=0,\ 2z+2t=0\}\\ &= \{(x,-x,z,-z) \mid x,z\in\mathbb{R}\}.\end{aligned}$$

As $\operatorname{Ker} f \neq \{\mathbf{0}\}$ we see that f is **not** injective.
A basis for $\operatorname{Ker} f$ is given by $\{(1,-1,0,0),(0,0,1,-1)\}$. This set is spanning as

$$(x,-x,z,-z) = x(1,-1,0,0) + z(0,0,1,-1) \qquad \forall x,z\in\mathbb{R}.$$

Moreover, as this set contains two vectors which are not multiples of each other, it is linearly independent, hence a basis for $\operatorname{Ker} f$. In particular, $\dim \operatorname{Ker} f = 2$.

Now using the Rank-Nullity Theorem we get

$$\dim \mathbb{R}^4 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

and as $\dim \mathbb{R}^4 = 4$ and $\dim \operatorname{Ker} f = 2$ we must have $\dim \operatorname{Im} f = 2$. As $\operatorname{Im} f$ is a subspace of $\mathbb{R}^2$ and has dimension 2 we must have $\operatorname{Im} f = \mathbb{R}^2$. In particular, f is surjective. As a basis for $\operatorname{Im} f = \mathbb{R}^2$ we could take the standard basis $\{(1,0),(0,1)\}$.

2. (a) We have $g(1) = 1+0 = 1 = 1.1 + 0x + 0x^2$, $g(x) = 1+1 = 2 = 2.1 + 0x + 0x^2$ and $g(x^2) = 1+2x = 1.1 + 2x + 0x^2$. Thus the matrix representing g with respect to the basis $\{1,x,x^2\}$ is given by

$$A = \begin{pmatrix} 1 & 2 & 1\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}.$$

(b) We have $g(1) = 1 = 1.1+0(1+x)+0(1+x+x^2)$, $g(1+x) = 3 = 3.1+0(1+x)+0(1+x+x^2)$ and $g(1+x+x^2) = 4+2x = 2.1+2(1+x)+0(1+x+x^2)$. Thus the matrix representing g with respect to the basis $\{1, 1+x, 1+x+x^2\}$ is given by

$$B = \begin{pmatrix} 1 & 3 & 2\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}.$$

(c) We have $1 = 1.1 + 0x + 0x^2$, $1+x = 1.1 + 1x + 0x^2$ and $1+x+x^2 = 1.1 + 1x + 1x^2$. Thus the change of basis matrix is given by

$$P = \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

We have

$$P^{-1} = \begin{pmatrix} 1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$$

and we check that

$$
\begin{aligned}
P^{-1}AP &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = B
\end{aligned}
$$

as required.

3.

$$\Delta(\lambda) = \det \begin{pmatrix} 2-\lambda & 0 & -3 \\ -6 & -1-\lambda & 6 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (\lambda+1)^2(2-\lambda).$$

Thus the eigenvalues of A are given by $\lambda = -1$ and $\lambda = 2$.

When $\lambda = -1$, consider

$$\begin{pmatrix} 3 & 0 & -3 \\ -6 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to $x - z = 0$, so $z = x$ and y can be anything. Thus we have

$$s_A(-1) = \{(x, y, z) \in \mathbb{R}^3 \mid z = x\} = \{(x, y, x) \mid x, y \in \mathbb{R}\}.$$

A basis for $s_A(-1)$ is given by $\{(1, 0, 1), (0, 1, 0)\}$. This set is a spanning set for $s_A(-1)$ as

$$(x, y, x) = x(1, 0, 1) + y(0, 1, 0) \qquad \forall x, y \in \mathbb{R}.$$

Moreover it is also linearly independent as it contains two vectors which are not multiples of each other.

When $\lambda = 2$, consider

$$\begin{pmatrix} 0 & 0 & -3 \\ -6 & -3 & 6 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to $z = 0$ and $-2x - y + 2z = 0$. Thus we have $z = 0$ and $y = -2x$ and

$$s_A(2) = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0,\ y = -2x\} = \{(x, -2x, 0) \mid x \in \mathbb{R}\}.$$

A basis for $s_A(2)$ is given by $\{(1, -2, 0)\}$. This set is a spanning set for $s_A(2)$ as

$$(x, -2x, 0) = x(1, -2, 0) \qquad \forall x \in \mathbb{R}.$$

Moreover it is also linearly independent as it contains only one non-zero vector.

Now we set

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

then P^{-1} is given by

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

and we have

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -3 \\ -6 & -1 & 6 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

a diagonal matrix with the eigenvalues appearing on the diagonal.

4. (a) As $\operatorname{rank} h_1 = \dim \operatorname{Im} h_1 = 3$ and $\operatorname{Im} h_1$ is a subspace of $\mathbb{R}^3$ we see that $\operatorname{Im} h_1 = \mathbb{R}^3$ and so h_1 is surjective.
Now using the Rank-Nullity theorem we have

$$\dim \mathbb{R}^4 = \operatorname{rank} h_1 + \operatorname{nullity} h_1$$

and as $\dim \mathbb{R}^4 = 4$ and $\operatorname{rank} h_1 = 3$ we get that $\operatorname{nullity} h_1 = \dim \operatorname{Ker} h_1 = 1$. This implies that h_1 is not injective.

(b) As 0 is an eigenvalue for h_2, there is a non-zero vector $\mathbf{v} \in V$ with $h_2(\mathbf{v}) = 0.\mathbf{v} = \mathbf{0}$, i.e. $\mathbf{v} \in \operatorname{Ker} h_2$. This implies that h_2 is not injective.
Now using the Rank-Nullity theorem we have

$$\dim V = \dim \operatorname{Ker} h_2 + \dim \operatorname{Im} h_2.$$

As $\dim \operatorname{Ker} h_2 > 1$ we have that $\dim \operatorname{Im} h_2 < \dim V$. So $\operatorname{Im} h_2 \neq V$ and h_2 is not surjective.