Linear Algebra: Solutions to Exercise Sheet 1

Remark on notation:
"$\{X\}$" means "the set of all X's".
"$\{X \mid Y\}$" means "the set of all X's satisfying condition Y".
"$\forall X$" means "for all X's".
"$x \in Y$" means "x belongs to the set Y.
"$x \notin Y$" means "x does not belong to the set Y.

1. P_n is the set of all polynomials $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ with $a_0, a_1, \ldots, a_n \in \mathbb{R}$, with addition given by

   $$(a_0+a_1x+\ldots+a_nx^n)+(b_0+b_1x+\ldots+b_nx^n) = (a_0+b_0)+(a_1+b_1)x+\ldots+(a_n+b_n)x^n \in P_n,$$

   and scalar multiplication given by

   $$\lambda(a_0 + a_1 x + \ldots + a_n x^n) = (\lambda a_0) + (\lambda a_1)x + \ldots + (\lambda a_n)x^n \in P_n$$

   for $\lambda \in \mathbb{R}$.

   We need to check that $(V1)$–$(V8)$ are satisfied.
   $(V1)$: $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ for all $p(x), q(x), r(x) \in P_n$ is clear.
   $(V2)$: $p(x) + q(x) = q(x) + p(x)$ for all $p(x), q(x) \in P_n$ is also clear.
   $(V3)$: The zero vector is given by the zero polynomial $0 = 0 + 0x + \ldots + 0x^n$.
   $(V4)$: For each $p(x) \in P_n$, its negative is given by $-p(x) = (-a_0) + (-a_1)x + \ldots + (-a_n)x^n$.
   $(V5)$: $\lambda(p(x) + q(x)) = (\lambda p(x)) + (\lambda q(x))$ for all $\lambda \in \mathbb{R}$ and all $p(x), q(x) \in P_n$.
   $(V6)$–$(V8)$: are similar.

2. $M(2,2)$ is the set of all 2×2 matrices with real entries with addition given by

   $$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

   and scalar multiplication given by

   $$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

   for $\lambda \in \mathbb{R}$.

   We need to check the 8 axioms.
   $(V1)$: For all matrices $A, B, C \in M(2,2)$ we have $(A + B) + C = A + (B + C)$.
   $(V2)$: For all $A, B \in M(2,2)$ we have $A + B = B + A$.
   $(V3)$: The zero vector is given by the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
   $(V4)$: For any matrix A, its negative is given by $-A = (-1)A$.

$(V5)$: $\lambda(A+B) = (\lambda A) + (\lambda B)$ for any $A, B \in M(2,2)$ and any $\lambda \in \mathbb{R}$.
$(V6)$–$(V8)$: can be verified similarly.

Important remark: We can generalize the above argument to show that in fact the set $M(n,m)$ of all $n \times m$ matrices with real entries, with the usual addition and scalar multiplication of matrices, form a vector space over $\mathbb{R}$.

3. (a) $U = \{(x,0) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. In order to prove it, we need to show that U satisfies the three conditions $(S1)$, $(S2)$ and $(S3)$.
$(S1)$: $(0,0) \in U$ (take $x = 0$).
$(S2)$: If $(x,0) \in U$ and $(y,0) \in U$ then $(x,0) + (y,0) = (x+y,0) \in U$.
$(S3)$: If $\lambda \in \mathbb{R}$ and $(x,0) \in U$ then $\lambda(x,0) = (\lambda x, 0) \in U$.

   (b) $U = \{(x,y) \mid x, y \text{ integers}\}$ is **not** a subspace of $\mathbb{R}^2$. In fact, $(S3)$ fails: take $\lambda = \frac{1}{2}$ and $(1,0) \in U$ then $\frac{1}{2}(1,0) = (\frac{1}{2}, 0) \notin U$.

   (c) $U = \{(x,y) \in \mathbb{R}^2 \mid x \leq y\}$ is **not** a subspace of $\mathbb{R}^2$. In fact, $(S3)$ fails: take $\lambda = -1$ and $(1,2) \in U$ then $(-1)(1,2) = (-1,-2) \notin U$ (as $-1 > -2$).

   (d) $U = \{(x,y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ is **not** a subspace of $\mathbb{R}^2$. In fact, $(S2)$ fails: take $(1,-1) \in U$ and $(1,1) \in U$ then $(1,-1) + (1,1) = (2,0) \notin U$ as $2^2 = 4 \neq 0$.

   (e) $U = \{(x,y) \in \mathbb{R}^2 \mid y = 2x\}$ is a subspace of $\mathbb{R}^2$. In order to prove it, we need to show that U satisfies the three conditions $(S1)$, $(S2)$ and $(S3)$.
$(S1)$: $(0,0) \in U$ as $0 = 2.0$.
$(S2)$: If $(x_1,y_1) \in U$ (i.e. $y_1 = 2x_1$) and $(x_2,y_2) \in U$ (i.e. $y_2 = 2x_2$) then $(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2) \in U$ as $y_1 + y_2 = 2x_1 + 2x_2 = 2(x_1+x_2)$.
$(S3)$: If $\lambda \in \mathbb{R}$ and $(x,y) \in U$ (i.e. $y = 2x$) then $\lambda(x,y) = (\lambda x, \lambda y) \in U$ as $\lambda y = \lambda(2x) = 2(\lambda x)$.

4. (a) $U = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = f(1)\}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. In order to prove it we need to verify that U satisfies the following three conditions.
$(S1)$: The zero function O defined by $O(x) = 0$ for all $x \in \mathbb{R}$ is in U as $O(0) = 0 = O(1)$.
$(S2)$: If $f, g \in U$ i.e. $f(0) = f(1)$ and $g(0) = g(1)$ then $(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1)$ so $(f+g) \in U$.
$(S3)$: If $\lambda \in \mathbb{R}$ and $f \in U$ i.e. $f(0) = f(1)$ then $(\lambda f)(0) = \lambda(f(0)) = \lambda(f(1)) = (\lambda f)(1)$. So $(\lambda f) \in U$.

   (b) $U = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) \geq 0 \quad \forall x \in \mathbb{R}\}$ is **not** a subspace of $\mathbb{R}^{\mathbb{R}}$. Indeed, condition $(S3)$ fails: take $f \in U$ defined by $f(x) = 1$ for all $x \in \mathbb{R}$, then $(-1)f \notin U$ as $((-1)f)(x) = -1 < 0$ for all $x \in \mathbb{R}$.

   (c) $U = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) = f(-x) \quad \forall x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. As before, to prove this we need to verify the same three conditions.
$(S1)$: The zero function O is in U, as $O(x) = 0 = O(-x)$ for all $x \in \mathbb{R}$.
$(S2)$: If $f, g \in U$, i.e. $f(x) = f(-x)$ and $g(x) = g(-x)$ for all $x \in \mathbb{R}$, then $(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$ for all $x \in \mathbb{R}$. So $(f+g) \in U$.
$(S3)$: If $\lambda \in \mathbb{R}$ and $f \in U$, i.e. $f(x) = f(-x)$, then $(\lambda f)(x) = \lambda(f(x)) = \lambda(f(-x)) = (\lambda f)(-x)$ for all $x \in \mathbb{R}$. So $(\lambda f) \in U$.

5. (a) $U = \{a_0 + a_1x + \ldots + a_nx^n \in P_n \mid a_0 + a_1 + \ldots + a_n = 0\}$ is a subspace of P_n. In order to prove it, we need to show that U satisfies the three conditions $(S1)$, $(S2)$ and $(S3)$.
$(S1)$: the zero polynomial is in U (take $a_0 = a_1 = \ldots = a_n = 0$).
$(S2)$: If $a_0+a_1x+\ldots+a_nx^n \in U$ (i.e. $a_0+\ldots+a_n = 0$) and $b_0+b_1x+\ldots+b_nx^n \in U$ (i.e. $b_0+\ldots+b_n = 0$) then $(a_0+\ldots+a_nx^n)+(b_0+\ldots+b_nx^n) = (a_0+b_0)+\ldots+(a_n+b_n)x^n \in U$ as $(a_0+b_0)+\ldots+(a_n+b_n) = (a_0+\ldots+a_n)+(b_0+\ldots+b_n) = 0+0 = 0$.
$(S3)$: If $\lambda \in \mathbb{R}$ and $a_0 + \ldots + a_nx^n \in U$ then $\lambda(a_0 + \ldots + a_nx^n) = (\lambda a_0) + \ldots + (\lambda a_n)x^n \in U$ as $(\lambda a_0) + \ldots + (\lambda a_n) = \lambda(a_0 + \ldots + a_n) = \lambda.0 = 0$.

(b) The set of all polynomials of degree exactly n is **not** a subspace of P_n. Indeed, condition $(S1)$ is not satisfied as the zero polynomial 0 has degree 0, so it is not in this set. (Can also show that $(S2)$ is not satisfied, as $x + x^n$ and $2x - x^n$ are both polynomials of degree n, so they are in this set, but their sum $(x+x^n)+(2x-x^n) = 3x$ is not a polynomial of degree n.)

(c) Note that the set U of all polynomials in P_n satisfying $p(0) = 0$ is equal to the set of all polynomials of the form $a_1x + a_2x^2 + \ldots + a_nx^n$ (so the independent term $a_0 = 0$). We claim that U is a subspace of P_n. In order to prove it, we need to show that $(S1)$, $(S2)$ and $(S3)$ holds.
$(S1)$: The zero polynomial $0 = 0 + 0x + 0x^2 + \ldots + 0x^n$ is in U.
$(S2)$: Let $a_1x + a_2x^2 + \ldots + a_nx^n$ and $b_1x + b_2x^2 + \ldots + b_nx^n$ be two polynomials in U, then their sum $(a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$ is still in U.
$(S3)$: Let $\lambda \in \mathbb{R}$ and let $p(x) = a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial in U, then $\lambda p(x) = (\lambda a_1)x + (\lambda a_2)x^2 + \ldots + (\lambda a_n)x^n$ is still in U.

6. (a) Let U be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Then U is a subspace of $M(2, 2)$. In order to prove it we need to show that conditions $(S1)$, $(S2)$ and $(S3)$ holds.
$(S1)$: The zero matrix is in U (take $a = b = c = 0$).
$(S2)$: If $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ are in U then their sum $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{pmatrix}$ is in U.
$(S3)$: If $\lambda \in \mathbb{R}$ and $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is in U then $\lambda \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ 0 & \lambda c \end{pmatrix}$ is also in U.

(b) The set U of all matrices of the form $\begin{pmatrix} a & 1 \\ 0 & c \end{pmatrix}$ for some $a, c \in \mathbb{R}$ is not a subspace of $M(2, 2)$ as, for example, the zero matrix is not in U, so condition $(S1)$ is not satisfied. (can also show that condition $(S2)$ and $(S3)$ are not satisfied).

(c) This is not a subspace of $M(2, 2)$ as the zero matrix is not in this set, so $(S1)$ is not satisfied. (We could also show that $(S2)$ or $(S3)$ is not satisfied).

7. We proved at the lectures that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and similarly that $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. Now, is $\mathbb{C}^n$ a vector space over $\mathbb{R}$? The answer is yes. The addition

and the axioms $(V1)$–$(V4)$ are satisfied just as before. For the scalar multiplication we have that if $\lambda \in \mathbb{R}$ and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then $\lambda\mathbf{z} = (\lambda z_1, \ldots, \lambda z_n) \in \mathbb{C}^n$ as $\lambda z_i \in \mathbb{C}$ for $1 \leq i \leq n$. Moreover, the axioms $(V5)$–$(V8)$ are satisfied as before.

Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$? The answer is no. The problem lies with the scalar multiplication. If $\lambda \in \mathbb{C}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\lambda\mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$ is not necessarily in $\mathbb{R}^n$. For example, take $n = 2$, $\lambda = i \in \mathbb{C}$ and $\mathbf{x} = (1, 1) \in \mathbb{R}^2$, then $\lambda\mathbf{x} = (i, i) \notin \mathbb{R}^2$.