Linear Algebra: Solutions to Exercise Sheet 2

1. (a) This set is both spanning and linearly independent, so it is a basis.
Spanning: Want to show that any vector in $\mathbb{R}^3$ can be written as a linear combination of these vectors. Take any vector $(a, b, c) \in \mathbb{R}^3$. We want to find $\lambda_1, \lambda_2, \lambda_3$ such that

$$\begin{aligned}(a,b,c) &= \lambda_1(1,0,0) + \lambda_2(5,5,0) + \lambda_3(3,3,3). \\ &= (\lambda_1 + 5\lambda_2 + 3\lambda_3, 5\lambda_2 + 3\lambda_3, 3\lambda_3).\end{aligned}$$

Thus we have to solve the following system of equations:

$$\begin{cases} a = \lambda_1 + 5\lambda_2 + 3\lambda_3 \\ b = 5\lambda_2 + 3\lambda_3 \\ c = 3\lambda_3 \end{cases}$$

This gives $\lambda_1 = a - b$, $\lambda_2 = \frac{b-c}{5}$ and $\lambda_3 = \frac{c}{3}$.
Linear independence: As the dimension of $\mathbb{R}^3$ is 3 and we have a spanning set of 3 vectors, this set is automatically linearly independent (use Corollary 1.24 from the lecture). So we don't need to check anything here. If you prefer not to use this result and check everything by hand, then you need to show that if

$$\lambda_1(1,0,0) + \lambda_2(5,5,0) + \lambda_3(3,3,3) = (0,0,0)$$

for some real numbers $\lambda_1, \lambda_2, \lambda_3$ then we must have that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. But the above vector equation is satisfied precisely when the following system of equations is satisfied:

$$\begin{cases} \lambda_1 + 5\lambda_2 + 3\lambda_3 = 0 \\ 5\lambda_2 + 3\lambda_3 = 0 \\ 3\lambda_3 = 0 \end{cases}$$

The only solution to this system of equations is $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

(b) This set cannot be spanning as the dimension of $\mathbb{R}^3$ is 3, so any spanning set has at least 3 vectors and this set has only 2 vectors. Is it linearly independent? Write $\lambda_1(2,5,4) + \lambda_2(-2,1,0) = (0,0,0)$. This implies that $(2\lambda_1 - 2\lambda_2, 5\lambda_1 + \lambda_2, 4\lambda_1) = (0,0,0)$, so $\lambda_1 = \lambda_2 = 0$. Hence this set is lineraly independent. It is not a basis as it is not spanning.

(c) This set is not linearly independent because it contains the zero vector. If we write

$$\lambda_1(-3,2,1) + \lambda_2(0,0,0) + \lambda_3(1,2,-3) = (0,0,0)$$

then this does not imply that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In fact λ_2 could be any real number. For example we could take $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = 5$ and the equation would be satisfied.

This set is not a spanning set for $\mathbb{R}^3$. We can either show this be seeing that this set contains at most 2 linearly independent vectors. So it spans a subspace of dimension at most 2. Otherwise, we can show this directly by trying to write any vector (a, b, c) in $\mathbb{R}^3$ as a linear combination of these three vectors i.e. we want to find real numbers λ_1, λ_2 and λ_3 such that

$$(a, b, c) = \lambda_1(-3, 2, 1) + \lambda_2(0, 0, 0) + \lambda_3(1, 2, -3).$$

This vector equation is satisfied precisely when the following system of equations is satisfied:

$$\begin{cases} a = -3\lambda_1 + \lambda_3 \\ b = 2\lambda_1 + 2\lambda_3 \\ c = \lambda_1 - 3\lambda_3 \end{cases}$$

From the first equation we have that $\lambda_3 = a + 3\lambda_1$. Plugging this into the second equation gives $8\lambda_1 = b - 2a$ and so $\lambda_1 = \frac{1}{8}(b - 2a)$. But plugging it into the third equation gives $-8\lambda_1 = c + 3a$ and so $\lambda_1 = -\frac{1}{8}(c + 3a)$. These can not be satisfied at the same time in general. Take for example $a = c = 0$ and $b = 8$ then we must have at the same time $\lambda_1 = 1$ and $\lambda_1 = 0$! This shows that the vector $(a, b, c) = (0, 8, 0)$ cannot be written as a linear combination of vectors in our set. Thus this set is not spanning.

(d) Linear independence? Write $\lambda_1(1, 1, 1) + \lambda_2(6, 2, 4) + \lambda_3(7, 3, 5) = (0, 0, 0)$. Does this imply that $\lambda_1 = \lambda_2 = \lambda_3 = 0$? Well the above vector equation is satisfied precisely when the following system of equations is satisfied:

$$\begin{cases} \lambda_1 + 6\lambda_2 + 7\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \\ \lambda_1 + 4\lambda_2 + 5\lambda_3 = 0 \end{cases}$$

The solution is $\lambda_1 = \lambda_2 = -\lambda_3$, so we could take $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 1$. Thus this set is not linearly independent.
Spanning? This set has at most two linearly independent vectors so it cannot be spanning. (can also check this directly if you prefer, just as in (c) above).

(e) This set cannot be linearly independent and it contains 4 vectors and the dimension of $\mathbb{R}^3$ is 3. Is it spanning? We need to check whether every vector $(a, b, c) \in \mathbb{R}^3$ can be written as a linear combination of these 4 vectors or not. Fix $a, b, c \in \mathbb{R}$, is it possible to find some $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (in terms of a, b, c) such that

$$(a, b, c) = \lambda_1(1, 0, 0) + \lambda_2(1, 1, 0) + \lambda_3(1, 1, 1) + \lambda_4(-1, 5, 2)$$

This is equivalent to the following system of linear equations

$$\begin{cases} a = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 \\ b = \lambda_2 + \lambda_3 + 5\lambda_4 \\ c = \lambda_3 + 2\lambda_4 \end{cases}$$

and we get $\lambda_1 = a - b + 6\lambda_4$, $\lambda_2 = b - c - 3\lambda_4$, $\lambda_3 = c - 2\lambda_4$ and λ_4 can be anything. We could for example take $\lambda_4 = 0$, $\lambda_1 = a - b$, $\lambda_2 = b - c$ and $\lambda_3 = c$. So this set is a spanning set for $\mathbb{R}^3$.

(f) This set cannot be linearly independent as it contains 4 vectors and the dimension of $\mathbb{R}^3$ is 3. In fact every vector in this set is a multiple of $(1, 0, 0)$. This set is not spanning either as any linear combination of vectors in this set will be of the form $(a, 0, 0)$ for some $a \in \mathbb{R}$. In particular, the vector $(1, 1, 1)$ cannot be written as a linear combination of vectors in this set.

2. (a) Linear independence? Write $\lambda_1 1 + \lambda_2(1 + x) + \lambda_3(1 + x + x^2) = 0$ (the zero polynomial). Then we have $(\lambda_1 + \lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3)x + \lambda_3 x^2 = 0$. This implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. So this set is linearly independent.
Spanning? We have a set containing 3 linearly independent vectors, so it is a basis (using Corollary 1.24), in particular, it is a spanning set. You can also prove this directly by showing that every polynomial $a_0 + a_1 x + a_2 x^2$ can be written as $\lambda_0 1 + \lambda_1(1 + x) + \lambda_2(1 + x + x^2)$ for some real numbers λ_0, λ_1 and λ_2. So you should find λ_0, λ_1 and λ_2 in terms of a_0, a_1 and a_2 (answer is $\lambda_0 = a_0 - a_1$, $\lambda_1 = a_1 - a_2$ and $\lambda_2 = a_2$).

(b) Linear independence? Note that $-1 + x + 2x^2 = 2 + 2x + x^2 - (3 + x - x^2)$ so this set is not linearly independent. If you cannot see it directly, use the same method as before. Write a linear combination of these vectors equals to the zero polynomial and show that it does not imply that all the coefficents must be zero.
Spanning? As before, as this set contains at most two linearly independent vectors, this cannot be a spanning set (as the dimension of P_2 is equal to 3).

(c) Linear independence? This set cannot be linearly independent as it contains 4 vectors and the dimension of P_2 is equal to 3. You can also see this directly as $1 + x + x^2 = 1 + x + \frac{1}{14}(14x^2)$.
Spanning? This set is spanning. Take any polynomial $a_0 + a_1 x + a_2 x^2$ in P_2, we want to find $\lambda_1, \lambda_2, \lambda_3$ and λ_4 such that $a_0 + a_1 x + a_2 x^2 = \lambda_1(1 + x + x^2) + \lambda_2 1 + \lambda_3 x + \lambda_4(14x^2)$. We can take $\lambda_1 = 0$, $\lambda_2 = a_0$, $\lambda_3 = a_1$ and $\lambda_4 = \frac{a_2}{14}$. Note that this is not unique as this set is not linearly independent.

3. We want to prove that $\{f, g, h\}$ is a linearly independent set. As before, write $\lambda_1 f + \lambda_2 g + \lambda_3 h = O$ (the zero vector is the zero function in this case). This is an equality of functions, so it holds precisely when $(\lambda_1 f + \lambda_2 g + \lambda_3 h)(x) = O(x) = 0$ for all $x \in \mathbb{R}$. This means that $\lambda_1 \sin x + \lambda_2 \cos x + \lambda_3 x = 0$ for all $x \in \mathbb{R}$. In particular, it is true for $x = 0$, and in this case the equation becomes $\lambda_2 = 0$. It also holds for $x = \frac{\pi}{2}$, so we get $\lambda_1 + \lambda_3 \frac{\pi}{2} = 0$. It also holds when $x = \pi$, so we get $-\lambda_2 + \lambda_3 \pi = 0$. So we deduce that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
As f, g and h are linearly independent, they form a basis for the subspace spanned by this three vectors and so the dimension of this subspace is 3.

4. First note that a polynomial $p(x)$ in P_n satisfies $p(0) = 0$ if and only if $p(x) = a_1 x + a_2 x^2 + \ldots + a_n x^n$ (i.e. $a_0 = 0$). We claim that the set $\{x, x^2, \ldots, x^n\}$ is a basis for this subspace. Clearly, it is spanning as every polynomial in this subspace can be written as a linear combination of these vectors, just take the coefficients to be $a_1, a_2, \ldots, a_n$. We also need to check that this set is linearly independent. Write $\lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_n x^n = 0$ (the zero polynomial). Then this implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. So it is linearly

independent.
As we have found a basis for U containing n elements, we deduce that the dimension of U is n. (Recall: the dimension of P_n is $n+1$).

5. Note that we can rewrite the vector space V as

$$V = \{(x, y, x + y, x - y) \mid x, y \in \mathbb{R}\}.$$

From this we can easily write every vector in V as

$$(x, y, x + y, x - y) = x(1, 0, 1, 1) + y(0, 1, 1, -1).$$

In other words, every vector in V can be written as a linear combination of $(1, 0, 1, 1)$ and $(0, 1, 1, -1)$. Thus the set $\{(1, 0, 1, 1), (0, 1, 1, -1)\}$ is a spanning set for V. We need to check that it is also linearly independent. This is clear as there are only two vectors in this set and one is not a multiple of the other. (Alternatively, you can also check this using the method used in question 1). Note that this is just **a** basis for V but there are many other choices.
As we found a basis for V containing 2 elements, the dimension of V is 2.

6. We want to show that the vector space P of all polynomials (having any degree) cannot be spanned by a finite number of vectors. Suppose, for a contradiction, that there is a finite spanning set $\{p_1(x), p_2(x), \ldots, p_k(x)\}$ where the $p_i(x)$'s are some polynomials. Let m be the maximal degree of the polynomials $p_i(x)$'s in the set. Then the polynomial x^{m+1} cannot be spanned by this set of vectors as taking linear combinations of polynomials never increases the degree. Thus this set is not spanning, but this is a contradiction. So we cannot have a finite spanning set for P.