Linear Algebra: Solutions to Exercise Sheet 3

1. Let V and W be vector spaces over $\mathbb{R}$. Recall that a map $f : V \longrightarrow W$ is linear if and only if the following two conditions hold:

   (i) $f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

   (ii) $f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$

   So in order to prove that a map *is* linear, we need to check that the two conditions given above hold. In order to prove that a map is *not* linear, it is enough to show that one of these conditions fails, i.e. to find particular vectors $\mathbf{v}$ and $\mathbf{u}$ which do not satisfy condition (i), or to find a particular value for λ and a particular vector $\mathbf{v}$ which do not satisfy condition (ii).

   (a) This map is linear.
   (i) $f((x,y)+(x',y')) = f(x+x', y+y') = 3(x+x') + 2(y+y') = 3x + 3x' + 2y + 2y' = (3x+2y) + (3x'+2y') = f(x,y) + f(x',y')$ for all $(x,y), (x',y') \in \mathbb{R}^2$.
   (ii) $f(\lambda(x,y)) = f(\lambda x, \lambda y) = 3(\lambda x) + 2(\lambda y) = \lambda(3x+2y) = \lambda f(x,y)$ for all $\lambda \in \mathbb{R}$ and for all $(x,y) \in \mathbb{R}^2$.

   (b) This map is not linear. Let's see that (ii) fails. Take $(x,y) = (1,1)$ and $\lambda = -1$. Then $f(-1(1,1)) = f(-1,-1) = (1,0)$ but $(-1)f(1,1) = (-1)(1,0) = (-1,0)$.

   (c) This map is linear.
   (i) $f(p(x)+q(x)) = (x+1)(p(x)+q(x)) = (x+1)p(x) + (x+1)q(x) = f(p(x)) + f(q(x))$ for all $p(x), q(x) \in P_n$.
   (ii) $f(\lambda p(x)) = (x+1)(\lambda p(x)) = \lambda(x+1)p(x) = \lambda f(p(x))$ for all $\lambda \in \mathbb{R}$ and all $p(x) \in P_n$.

   (d) This map is linear.
   (i) $f(p(x)+q(x)) = \int_0^1 (p(x)+q(x))dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = f(p(x)) + f(q(x))$ for all $p(x), q(x) \in P_n$.
   (ii) $f(\lambda p(x)) = \int_0^1 (\lambda p(x))dx = \lambda \int_0^1 p(x)dx = \lambda f(p(x))$ for all $\lambda \in \mathbb{R}$ and all $p(x) \in P_n$.

   (e) This map is not linear. We show that (i) fails. Take $p(x) = x$ and $q(x) = x+1$ then $f(p(x)+q(x)) = \frac{d}{dx}(2x+1) + (5x+2) = 5x+4$. But we have $f(p(x)) + f(q(x)) = \frac{d}{dx}(x) + (5x+2) + \frac{d}{dx}(x+1) + (5x+2) = 10x+6$. We could also have shown that condition (ii) fails.

   (f) This map is linear.
   (i)

$$
\begin{aligned}
f((x,y,z)+(x',y',z')) &= f(x+x', y+y', z+z') \\
&= ((y+y')+(z+z'), (x+x')+(z+z'), (x+x')+(y+y')) \\
&= ((y+z)+(y'+z'), (x+z)+(x'+z'), (x+y)+(x'+y')) \\
&= (y+z, x+z, x+y) + (y'+z', x'+z', x'+y') \\
&= f(x,y,z) + f(x',y',z')
\end{aligned}
$$

   for all $(x,y,z), (x',y',z') \in \mathbb{R}^3$
   (ii) $f(\lambda(x,y,z)) = f(\lambda x, \lambda y, \lambda z) = (\lambda y + \lambda z, \lambda x + \lambda z, \lambda x + \lambda y) = \lambda(y+z, x+z, x+y) = \lambda f(x,y,z)$ for all $\lambda \in \mathbb{R}$ and all $(x,y,z) \in \mathbb{R}^3$.

(g) This is a linear map.
(i) $f(A+B) = (A+B)^T = A^T + B^T = f(A) + f(B)$ for all $A, B \in M(n,m)$.
(ii) $f(\lambda A) = (\lambda A)^T = \lambda A^T$ for all $\lambda \in \mathbb{R}$ and all $A \in M(n,m)$.

(h) This map is not linear. Let's show that (i) fails. Take $n = 2$ and take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity 2×2 matrix. Then $f(A+A) = f\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$. But $f(A) + f(A) = det(A) + det(A) = 1 + 1 = 2$.

2. No, there is no such map. To see that observe that $(3,1) = 2(1,0) + (1,1)$ and so if the map f were to be linear then we must have $f(3,1) = f(2(1,0)) + f(1,1) = 2f(1,0) + f(1,1) = 2(3,2,1) + (-1,0,1) = (5,4,3)$ but here we are given $f(3,1) = (5,0,-2)$, so this map cannot be linear.

   Note that if the question read, "Is there a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(1,0) = (3,2,1)$, $f(1,1) = (-1,0,1)$ and $f(3,1) = (5,4,3)$?" then the answer would be yes. Indeed this follows from Proposition 2.3 in the lecture as $\{(1,0),(1,1)\}$ form a basis for $\mathbb{R}^2$ (easy to check), so there is a unique linear map f such that $f(1,0) = (3,2,1)$ and $f(1,1) = (-1,0,1)$. Moreover, this map satisfies
$$f(\lambda_1(1,0) + \lambda_2(1,1)) = \lambda_1 f(1,0) + \lambda_2 f(1,1)$$
   for any $\lambda_1, \lambda_2 \in \mathbb{R}$. So we have
$$f(3,1) = f(2(1,0) + (1,1)) = 2(3,2,1) + (-1,0,1) = (5,4,3).$$

3. A linear map $f : V \to W$ is surjective if for every vector $\mathbf{w} \in W$ there is a vector $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \mathbf{w}$. So a map is surjective precisely when the image of f, $\operatorname{Im} f$, is equal to W.

   We say that the map f is injective if distinct vectors in V are mapped to distinct vectors in W, in other words if whenever we have $f(\mathbf{v}) = f(\mathbf{u})$ for some $\mathbf{u}, \mathbf{v} \in V$ then we must have $\mathbf{v} = \mathbf{u}$. We have seen in the lectures, Proposition 2.10, that a linear map f is injective if and only if $\operatorname{Ker} f = \{\mathbf{0}\}$. So in order to prove that a linear map is injective, you can either show that the first condition holds, or prove that the kernel contains only the zero vector.

   (a) This map is not injective as $f(1,0,0) = f(0,1,0)$ but $(1,0,0) \neq (0,1,0)$. Alternatively, we see that $f(1,-1,0) = (0,0)$, so $(1,-1,0) \in \operatorname{Ker} f$.
   This map is surjective as for all $(x,y) \in \mathbb{R}^2$ we can find $(x,0,y) \in \mathbb{R}^3$ such that $f(x,0,y) = (x,y)$.

   (b) This map is injective. In order to prove it we check that the kernel is zero. Let $(x,y,z) \in \operatorname{Ker} f$, this means that $f(x,y,z) = (y+z, x+z, x+y) = (0,0,0)$ and so $x = y = z = 0$, thus $(x,y,z) = (0,0,0)$.
   This map is also surjective. For all $(x', y', z') \in \mathbb{R}^3$, we need to find $(x,y,z) \in \mathbb{R}^3$ such that $f(x,y,z) = (y+z, x+z, x+y) = (x', y', z')$. This can can done by taking $x = \frac{1}{2}(-x' + y' + z')$, $y = \frac{1}{2}(x' - y' + z')$ and $z = \frac{1}{2}(x' + y' - z')$.

   (c) This map is not injective as $f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$.
   This map is surjective as for all $x \in \mathbb{R}$ we have $f\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = x$.

(d) This map is injective. We show that the kernel is zero. Suppose $f(a_0 + a_1x) = a_0x + a_1x^2 = 0$ then $a_0 = a_1 = 0$ so $a_0 + a_1x = 0$.
This map is not surjective as the constant polynomial 1 is not in the image of f.

(e) This map is not injective as $f(0, 0, 0) = f(1, 0, 1) = (0, 0, 0)$.
This map is not surjective as $(1, 2, 3)$ is not in the image.

(f) Note that for $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in P_n$, we have $p(0) = a_0$. So the map f can be defined as $f(a_0 + a_1x + a_2x^2 + \ldots a_nx^n) = a_1x + a_2x^2 + \ldots + a_nx^n$.
This map is not injective as $f(x + 2) = f(x) = x$.
This map is not surjective as the constant polynomial 1 is not in the image.

4. (a) The Rank-Nullity theorem tells us that

$$\dim \mathbb{R}^3 = \dim \ker f + \dim \mathrm{Im} f$$

But we have seen above that the map f is surjective, so $\dim \mathrm{Im} f = 2$. Hence we have $\dim \ker f = 1$. It is easy to find a basis for $\mathrm{Im} f = \mathbb{R}^2$, take for example $\{(1, 0), (0, 1)\}$. Now $\ker f = \{(x, y, z) \in \mathbb{R}^3 \mid (x + y, z) = (0, 0)\} = \{(x, -x, 0) \mid x \in \mathbb{R}\}$. So $\{(1, -1, 0)\}$ is certainly a linearly independent set in $\ker f$ and as $\dim \ker f = 1$ it is a basis.

(b) We have shown above that this map is both surjective and injective, so $\ker f = \{\mathbf{0}\}$ has no basis and $\mathrm{Im} f = \mathbb{R}^3$ has a basis given by $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for example.

(c) The Rank-Nullity theorem tells us that

$$\dim M(2, 2) = \dim \ker f + \dim \mathrm{Im} f$$

We have seen that the map f is surjective, so $\mathrm{Im} f = \mathbb{R}$ and $\dim \mathrm{Im} f = 1$. Moreover, we have $\dim M(2, 2) = 4$. Hence we must have $\dim \ker f = 3$. A basis for $\mathrm{Im} f = \mathbb{R}$ is given by $\{1\}$. The kernel of f is defined by

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b + c + d = 0\} = \{\begin{pmatrix} a & b \\ c & -a - b - c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}.$$

It is easy to check that $\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} & 0 \\ 1 & -1 \end{pmatrix}\}$ is linearly independent and as $\dim \ker f = 3$ it must be a basis.

(d) The Rank-Nullity theorem tells us that

$$\dim P_1 = \dim \mathrm{Im} f + \dim \ker f$$

We have seen above that f is injective, so $\dim \ker f = 0$ and $\ker f$ has no basis. As $\dim P_1 = 2$, we must have $\dim \mathrm{Im} f = 2$. Now $\mathrm{Im} f = \{a_0x + a_1x^2 \mid a_0, a_1 \in \mathbb{R}\}$. Clearly x and x^2 are in this set and they are linearly independent, as $\dim \mathrm{Im} f = 2$, the set $\{x, x^2\}$ must be a basis.

(e) The Rank-Nullity theorem tells us that

$$\dim \mathbb{R}^3 = \dim \mathrm{Im} f + \dim \ker f$$

The image of f is given by $\{(y, y, y) \mid y \in \mathbb{R}\}$, so it is easy to see that $\{(1, 1, 1)\}$ is both spanning and linearly independent, hence it is a basis for $\mathrm{Im} f$ and $\dim \mathrm{Im} f = 1$. Thus using the formula above we see that $\dim \ker f = 2$. Now $\ker f = \{(x, 0, z) \mid x, y \in \mathbb{R}\}$. Clearly $(1, 0, 0)$ and $(0, 0, 1)$ are linearly independent vectors in $\ker f$ and so $\{(1, 0, 0), (0, 0, 1)\}$ form a basis for $\ker f$.

(f) The Rank Nullity theorem tells us that

$$\dim P_n = \dim \mathrm{Im} f + \dim \ker f$$

Now $\mathrm{Im} f = \{a_1 x + a_2 x^2 + \ldots + a_n x^n \mid a_1, a_2, \ldots a_n \in \mathbb{R}\}$, so a basis for $\mathrm{Im} f$ is given by $\{x, x^2, \ldots x^n\}$ (check). As $\dim P_n = n+1$ and $\dim \mathrm{Im} f = n$, we must have $\dim \ker f = 1$. Clearly, the constant polynomial 1 is in the kernel so it is a basis for the kernel.

5. There are of course many different answers to this question. Here is one example.

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (0, 0)$. This is the zero map, it sends every vector to the zero vector. So the kernel $\mathrm{Ker}\, f = \mathbb{R}^2$ and the image $\mathrm{Im}\, f = \{\mathbf{0}\}$. Thus in this case we have $\mathrm{Im}\, f \subsetneq \mathrm{Ker}\, f$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, y)$. This is the identity map, it send every vector to itself. So the image $\mathrm{Im}\, f = \mathbb{R}^2$ and the kernel $\mathrm{Ker}\, f = \{\mathbf{0}\}$. Thus in this case we have $\mathrm{Ker}\, f \subsetneq \mathrm{Im}\, f$.

   (c) This case is slightly harder. First note that we must have $f \circ f = O$ the zero map, as everything in the image of f is then mapped to zero when we apply f a second time. We can take the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (0, x)$. It is easy to check that this map is linear. The kernel of f is given by $\mathrm{Ker}\, f = \{(0, y) \in \mathbb{R}^2\}$ and the image of f is given by $\{(0, x) \in \mathbb{R}^2\}$. Thus the image and the kernel of f coincide.

6. Let $\mathbf{u} \in \ker f$, this means that $f(\mathbf{u}) = \mathbf{0}$. Then $(g \circ f)(\mathbf{u}) = g(f(\mathbf{u})) = g(\mathbf{0}) = \mathbf{0}$. So $\mathbf{u} \in \ker(g \circ f)$. We have shown that $\ker f \subseteq \ker(g \circ f)$, thus $\dim \ker(g \circ f) \geq \dim \ker f$, i.e. $\mathrm{Nullity}(g \circ f) \geq \mathrm{Nullity}(f)$. Now the Rank-Nullity theorem applied to the map f tells us that

$$\dim U = \mathrm{Nullity}(f) + \mathrm{Rank}(f)$$

and the Rank-Nullity theorem applied to the map $(g \circ f)$ tells us that

$$\dim U = \mathrm{Nullity}(g \circ f) + \mathrm{Rank}(g \circ f).$$

As we have seen above that $\mathrm{Nullity}(g \circ f) \geq \mathrm{Nullity}(f)$, we must have $\mathrm{Rank}(g \circ f) \leq \mathrm{Rank}(f)$.