Linear Algebra: Solutions to Exercise Sheet 4

1. (a) We need to find a, b, c such that $7\mathbf{e_1} + 5\mathbf{e_2} - \mathbf{e_3} = a\mathbf{e_1} + b(\mathbf{e_1} + \mathbf{e_2}) + c(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$. So we need to solve the following system of equations:

$$\begin{cases} a + b + c = 7 \\ b + c = 5 \\ c = -1 \end{cases}$$

The solution is $a = 2$, $b = 6$ and $c = -1$. Thus the coordinate vector of $7\mathbf{e_1} + 5\mathbf{e_2} - \mathbf{e_3}$ with respect to the basis $\{\mathbf{e_1}, \mathbf{e_1} + \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}\}$ is $\begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix}$.

(b) We need to find a, b, c such that $5x^2 - 2x + 3 = a1 + b(1 + x) + c(1 + x^2) = (a + b + c)1 + bx + cx^2$. So we have $a = 0$, $b = -2$ and $c = 5$ and the corrdinate vector is $\begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$.

(c) We need to find a', b', c' in terms of a, b, c such that $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = a'(\mathbf{v_1} + \mathbf{v_2}) + b'(\mathbf{v_1} - \mathbf{v_2}) + c'\mathbf{v_3}$. So we must solve the following system of equations

$$\begin{cases} a' + b' = a \\ a' - b' = b \\ c' = c \end{cases}$$

The solution is $a' = \frac{a+b}{2}$, $b' = \frac{a-b}{2}$ and $c' = c$. Thus the coordinate vector is given by

$$\begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \\ c \end{pmatrix}$$

2. $f(\mathbf{e_1} - \mathbf{e_2}) = (2\mathbf{e_1} - \mathbf{e_3}) - (\mathbf{e_2} + \mathbf{e_3}) = 2\mathbf{e_1} - \mathbf{e_2} - 2\mathbf{e_3} = 3\mathbf{e_1} + (\mathbf{e_1} + \mathbf{e_2}) - 2(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$.
$f(\mathbf{e_1} + \mathbf{e_2}) = (2\mathbf{e_1} - \mathbf{e_3}) + (\mathbf{e_2} + \mathbf{e_3}) = 2\mathbf{e_1} + \mathbf{e_2} = \mathbf{e_1} + (\mathbf{e_1} + \mathbf{e_2})$.
Thus the matrix representing f with respect to the given bases is

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ -2 & 0 \end{pmatrix}$$

3. $f(\mathbf{e_1} - \mathbf{e_2}) = f(1, -1) = (1, -1, -1) = \mathbf{e_1} - \mathbf{e_2} - \mathbf{e_3}$
$f(\mathbf{e_1} + \mathbf{e_2}) = f(1, 1) = (1, 1, 1) = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$
Thus the matrix representing f with respect to the given bases is

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Now, $g(\mathbf{e_1}) = g(1, 0, 0) = (1, 0) = \mathbf{e_1} = \frac{1}{2}(\mathbf{e_1} - \mathbf{e_2}) + \frac{1}{2}(\mathbf{e_1} + \mathbf{e_2})$.
$g(\mathbf{e_2}) = g(0, 1, 0) = (1, 0) = \frac{1}{2}(\mathbf{e_1} - \mathbf{e_2}) + \frac{1}{2}(\mathbf{e_1} + \mathbf{e_2})$.

$g(\mathbf{e_3}) = g(0,0,1) = (0,1) = -\frac{1}{2}(\mathbf{e_1} - \mathbf{e_2}) + \frac{1}{2}(\mathbf{e_1} + \mathbf{e_2})$.
So the matrix representing g with respect to the given bases is

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Now consider the map $(g \circ f) : \mathbb{R}^2 \to \mathbb{R}^2$. It is given by $(g \circ f)(x,y) = (x+y, y)$.
$(g \circ f)(\mathbf{e_1} - \mathbf{e_2}) = (g \circ f)(1,-1) = (0,-1) = -\mathbf{e_2} = \frac{1}{2}(\mathbf{e_1} - \mathbf{e_2}) - \frac{1}{2}(\mathbf{e_1} + \mathbf{e_2})$.
$(g \circ f)(\mathbf{e_1} + \mathbf{e_2}) = (g \circ f)(1,1) = (2,1) = 2\mathbf{e_1} + \mathbf{e_2} = \frac{1}{2}(\mathbf{e_1} - \mathbf{e_2}) + \frac{3}{2}(\mathbf{e_1} + \mathbf{e_2})$.
Thus the matrix representing $(g \circ f)$ with respect to the given basis is

$$C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

We now check that the product rule holds in this case, namely that $C = BA$.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$$

4. $f(1) = 0$, $f(x) = 1$ and $f(x^2) = 2x$, so the matrix representing f with respect to the basis $\{1, x, x^2\}$ of P_2 is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Now, $f(1) = 0$, $f(1+x) = 1$ and $f(1+x^2) = 2x = -2.1 + 2(x+1)$, so the matrix representing f with respect to the basis $\{1, 1+x, 1+x^2\}$ of P_2 is given by

$$B = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We need to check that the change of basis theorem holds in this case. The change of basis matrix P (whose columns are the coordinate of the new basis vectors with respect to the old ones) is given by

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We check that $P^{-1}AP = B$,

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

5. Fix a basis $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ for V and a basis $\{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$ for W. The matrix $A = (a_{ij})$ representing the map f in these bases is defined by

$$f(\mathbf{v_i}) = \sum_{j=1}^{m} a_{ji} \mathbf{w_j}.$$

The matrix $B = (b_{ij})$ representing the map g in these bases is defined by

$$g(\mathbf{v_i}) = \sum_{j=1}^{m} b_{ji}\mathbf{w_j}.$$

We want to find the matrix representing $(f + g)$ in these bases. So we need to consider $(f + g)(\mathbf{v_i})$ and write it as a linear combination of the $\mathbf{w_j}$'s. We have

$$(f + g)(\mathbf{v_i}) = f(\mathbf{v_i}) + g(\mathbf{v_i}) = \sum_{j=1}^{m} a_{ji}\mathbf{w_j} + \sum_{j=1}^{m} b_{ji}\mathbf{w_j} = \sum_{j=1}^{m}(a_{ji} + b_{ji})\mathbf{w_j}.$$

Thus the matrix representing $(f + g)$ in these bases is given by $A + B$. Next we want to find the matrix representing (λf) in these bases. So we need to consider $(\lambda f)(\mathbf{v_i})$ and write it as a linear combination of the $\mathbf{w_j}$'s. We have

$$(\lambda f)(\mathbf{v_i}) = \lambda(f(\mathbf{v_i})) = \lambda\sum_{j=1}^{m} a_{ji}\mathbf{w_j} = \sum_{j=1}^{m}(\lambda a_{ji})\mathbf{w_j}.$$

Thus the matrix representing (λf) in these bases is given by λA.