Linear Algebra: Solutions to Exercise Sheet 5

1. (a) This is true. If f has eigenvalue λ, then there is some non-zero vector $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \lambda\mathbf{v}$. Now $f^{-1}(f(\mathbf{v})) = f^{-1}(\lambda\mathbf{v}) = \lambda f^{-1}(\mathbf{v})$ as f is linear. On the other hand, $f^{-1}(f(\mathbf{v})) = id(\mathbf{v}) = \mathbf{v}$. Thus we have $\lambda f^{-1}(\mathbf{v}) = \mathbf{v}$. As $\mathbf{v} \neq \mathbf{0}$, we must have $\lambda \neq 0$ and so we get $f^{-1}(\mathbf{v}) = \lambda^{-1}\mathbf{v}$ and $\mathbf{v}$ is an eigenvector of f^{-1} with eigenvalue λ^{-1}.

   (b) This is also true. If A has eigenvalue λ then there exists $\mathbf{x} = (x_1, \ldots, x_n) \neq (0, \ldots, 0)$ such that $A\mathbf{x} = \lambda\mathbf{x}$. But then $A^m\mathbf{x} = A^{m-1}(A\mathbf{x}) = A^{m-1}(\lambda\mathbf{x}) = \lambda A^{m-1}(\mathbf{x}) = \lambda A^{m-2}(A\mathbf{x}) = \lambda A^{m-2}(\lambda\mathbf{x}) = \lambda^2 A^{m-2}(\mathbf{x}) = \ldots = \lambda^m\mathbf{x}$. Thus λ^m is an eigenvalue of A^m.

2. (a) Consider the characteristic equation

$$\det\begin{pmatrix} 6-\lambda & 4 \\ -1 & 2-\lambda \end{pmatrix} = (\lambda-4)^2 = 0$$

   So the only eigenvalue is $\lambda = 4$. Consider the eigenspace $s_A(4)$:

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Thus we get $2x + 4y = 0$ (the other equation gives the same condition), so $x = -2y$ and $s_A(4) = \{(-2y, y) \mid y \in \mathbb{R}\}$. The subspace $s_A(4)$ has a basis given by $\{(-2, 1)\}$ (check). We cannot find a basis of eigenvectors for $\mathbb{R}^2$, so A is not diagonalizable.

   (b) Consider the characteristic equation

$$\det\begin{pmatrix} -2-\lambda & 0 \\ 6 & 1-\lambda \end{pmatrix} = (-2-\lambda)(1-\lambda) = 0$$

   So the eigenvalues are $\lambda = 1$ and $\lambda = -2$.
   When $\lambda = 1$ we get

$$\begin{pmatrix} -3 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Thus we get $x = 0$ and $s_A(1) = \{(0, y) \mid y \in \mathbb{R}\}$. It has a basis given by $\{(0, 1)\}$.
   When $\lambda = -2$ we get

$$\begin{pmatrix} 0 & 0 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Thus we get $6x + 3y = 0$ and $y = -2x$, so $s_A(-2) = \{(x, -2x) \mid x \in \mathbb{R}\}$. It has a basis given by $\{(1, -2)\}$.

   The change of basis matrix is given by $P = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and we have

$$P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

   .

(c) Consider the characteristic equation

$$\det\begin{pmatrix} -\lambda & 0 & 0 \\ 5 & 5-\lambda & 5 \\ 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(5-\lambda) = 0.$$

Thus the eigenvalues are $\lambda = 0$ and $\lambda = 5$.
When $\lambda = 0$ we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we get $5x + 5y + 5z = 0$, so $z = -x - y$ and $s_A(0) = \{(x, y, -x-y) \mid x, y \in \mathbb{R}\}$. The subspace $s_A(0)$ has a basis given by $\{(1, 0, -1), (0, 1, -1)\}$. It's easy to see that this set is spanning as $(x, y, -x-y) = x(1, 0, -1) + y(0, 1, -1)$ for all $x, y \in \mathbb{R}$ and it is linearly independent as it only contains two vectors and they are not multiple of each other.
When $\lambda = 5$ we have

$$\begin{pmatrix} -5 & 0 & 0 \\ 5 & 0 & 5 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we get $-5x = 0$, $5x + 5z = 0$ and $-5z = 0$, so $z = x = 0$ and $s_A(5) = \{(0, y, 0) \mid y \in \mathbb{R}\}$. The subspace $s_A(5)$ has a basis given by $\{(0, 1, 0)\}$. It's easy to see that this set is spanning as $(0, y, 0) = y(0, 1, 0)$ for all $y \in \mathbb{R}$ and it is linearly independent as it only contains one non-zero vector.

The change of basis matrix is given by $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$
and we have

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(d) Consider the characteristic equation

$$\det\begin{pmatrix} -\lambda & -1 & -3 \\ 2 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{pmatrix} = (\lambda-2)(-\lambda+4)(\lambda+2) = 0.$$

Thus the eigenvalues are $\lambda = 2$, $\lambda = 4$ and $\lambda = -2$.
When $\lambda = 2$ we have

$$\begin{pmatrix} -2 & -1 & -3 \\ 2 & 1 & 3 \\ -2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we get $-2x - y - 3z = 0$, $2x + y + 3z + 0$ and $-2x + y - z = 0$ so $y = -z$, $x = -z$ and $s_A(2) = \{(-z, -z, z) \mid z \in \mathbb{R}\}$. The subspace $s_A(2)$ has a basis given by $\{(-1, -1, 1)\}$. It's easy to see that this set is spanning as $(-z, -z, z) = z(-1, -1, 1)$ for all $z \in \mathbb{R}$ and it is linearly independent as it only contains one non-zero vector.

When $\lambda = -2$ we have

$$\begin{pmatrix} 2 & -1 & -3 \\ 2 & 5 & 3 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we get $2x - y - 3z = 0$ and $2x + 5y + 3z = 0$. Solving these equations we obtain $x = -y$ and $z = -y$. Thus the subspace $s_A(-2) = \{(-y, y, -y) \mid y \in \mathbb{R}\}$ and it has a basis given by $\{(-1, 1, -1)\}$ (check as before).
When $\lambda = 4$ we have

$$\begin{pmatrix} -4 & -1 & -3 \\ 2 & -1 & 3 \\ -2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we get $-4x - y - 3z = 0$ and $2x - y + 3z = 0$. Solving these two equations we get $x = -y$ and $z = y$. Thus the subspace $s_A(4)$ is given by $s_A(4) = \{(-y, y, y) \mid y \in \mathbb{R}\}$. This subspace has a basis given by $\{(-1, 1, 1)\}$ (check as before).
The change of basis matrix $P = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and we have

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

3. (a) $A = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix}$.

   (b) Tuesday:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.85 \\ 0.15 \end{pmatrix}$$

Wednesday:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix} \begin{pmatrix} 0.85 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 0.875 \\ 0.125 \end{pmatrix}$$

After n days:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix}^n \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$$

In order to calculate the n-th power of A we first need to diagonalize this matrix (if possible). We use the same method as in question 2. First consider the characteristic equation:

$$\det \begin{pmatrix} 0.95 - \lambda & 0.45 \\ 0.05 & 0.55 - \lambda \end{pmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5) = 0$$

So the eigenvalues of A are given by 1 and 0.5.
The eigenspace $s_A(1)$ corresponding to $\lambda = 1$ is given as

$$\begin{pmatrix} -0.05 & 0.45 \\ 0.05 & -0.45 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so we get

$$s_A(1) = \{(x, y) \in \mathbb{R}^2 \mid x = 9y\}.$$

A basis for $s_A(1)$ is given by $\{(9, 1)\}$.
Similarly, the eigenspace $s_A(0.5)$ corresponding to $\lambda = 0.5$ is given by

$$s_A(0.5) = \{(x, y) \in \mathbb{R}^2 \mid y = -x\}.$$

A basis for $s_A(0.5)$ is given by $\{(1, -1)\}$.
Thus we have $P = \begin{pmatrix} 9 & 1 \\ 1 & -1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & -0.9 \end{pmatrix}$ and

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} = D.$$

Going back to the original question we have

$$\begin{aligned} A^n &= PD^nP^{-1} = \begin{pmatrix} 9 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}^n \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & -0.9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.9 + 0.1(0.5)^n & 0.9 - 0.9(0.5)^n \\ 0.1 - 0.1(0.5)^n & 0.1 + 0.9(0.5)^n \end{pmatrix} \end{aligned}$$

Thus we have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.9 - 0.1(0.5)^n \\ 0.1 + 0.1(0.5)^n \end{pmatrix}.$$

And finally we can find the long term evolution of the distribution of healthy and ill students i.e. letting $n \to \infty$ we get

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}.$$

Thus in the long term, 90 percent of students will be healthy and 10 percent will be ill.