Linear Algebra: Solutions to Exercise Sheet 6

1. (a) This function is an inner product on $\mathbb{R}^3$. We need to check (i)–(iv).
(i) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ we have

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle &= 3x_1y_1 + 2x_2y_2 + 4x_3y_3 \\ &= 3y_1x_1 + 2y_2x_2 + 4y_3x_3 \\ &= \langle \mathbf{y}, \mathbf{x} \rangle.\end{aligned}$$

(ii) For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{R}^3$ we have

$$\begin{aligned}\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle &= 3(x_1 + x_1')y_1 + 2(x_2 + x_2')y_2 + 4(x_3 + x_3')y_3 \\ &= (3x_1y_1 + 2x_2y_2 + 4x_3y_3) + (3x_1'y_1 + 2x_2'y_2 + 4x_3'y_3) \\ &= \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle.\end{aligned}$$

(iii) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and all $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\langle \lambda\mathbf{x}, \mathbf{y} \rangle &= 3(\lambda x_1)y_1 + 2(\lambda x_2)y_2 + 4(\lambda x_3)y_3 \\ &= \lambda(3x_1y_1 + 2x_2y_2 + 4x_3y_3) \\ &= \lambda\langle \mathbf{x}, \mathbf{y} \rangle.\end{aligned}$$

(iv) For all $\mathbf{x} \in \mathbb{R}^3$ we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = 3x_1^2 + 2x_2^2 + 4x_3^2 \geq 0.$$

Moreover, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $x_1 = x_2 = x_3 = 0$, i.e. $\mathbf{x} = \mathbf{0}$.

(b) This function is not an inner product. We can see that (iv) fails: Take $\mathbf{x} = (0, 0, 1)$ then $\langle \mathbf{x}, \mathbf{x} \rangle = -6 < 0$.

(c) This function is not an inner product. We can see that (iii) fails: Take $\mathbf{x} = \mathbf{y} = (1, 1)$ and $\lambda = 2$ then $\langle \lambda\mathbf{x}, \mathbf{y} \rangle = \langle (2, 2), (1, 1) \rangle = 4$ but $\lambda\langle \mathbf{x}, \mathbf{y} \rangle = 2\langle (1, 1), (1, 1) \rangle = 2.1 = 2$.

(d) This is an inner product. We need to check that (i)–(iv) are satisfied.
(i) For all $p(x), q(x) \in P_2$ we have

$$\begin{aligned}\langle p(x), q(x) \rangle &= \int_{-1}^{1} p(x)q(x)dx \\ &= \int_{-1}^{1} q(x)p(x)dx \\ &= \langle q(x), p(x) \rangle.\end{aligned}$$

(ii) For all $p(x), q(x), r(x) \in P_2$ we have

$$\begin{aligned}\langle p(x) + q(x), r(x) \rangle &= \int_{-1}^{1} (p(x) + q(x))r(x)dx \\ &= \int_{-1}^{1} (p(x)r(x) + q(x)r(x))dx \\ &= \int_{-1}^{1} p(x)r(x)dx + \int_{-1}^{1} q(x)r(x)dx \\ &= \langle p(x), r(x) \rangle + \langle q(x), r(x) \rangle.\end{aligned}$$

(iii) For all $p(x), q(x) \in P_2$ and all $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \langle \lambda p(x), q(x) \rangle &= \int_{-1}^{1} (\lambda p(x)) q(x) dx \\ &= \int_{-1}^{1} \lambda (p(x) q(x)) dx \\ &= \lambda \int_{-1}^{1} p(x) q(x) dx \\ &= \lambda \langle p(x), q(x) \rangle. \end{aligned}$$

(iv) For all $p(x) \in P_2$ we have

$$\langle p(x), p(x) \rangle = \int_{-1}^{1} p(x)^2 dx \geq 0$$

as it is the integral of a non-negative function. Moreover $\langle p(x), p(x) \rangle = 0$ if and only if $p(x) = 0$ the zero polynomial.

(e) This is an inner product. We need to check that (i)–(iv) are satisfied.
(i) For all $p(x) = p_0 + p_1 x + p_2 x^2, q(x) = q_0 + q_1 x + q_2 x^2 \in P_2$ we have

$$\begin{aligned} \langle p(x), q(x) \rangle &= p_0 q_0 + p_1 q_1 + p_2 q_2 \\ &= q_0 p_0 + q_1 p_1 + q_2 p_2 \\ &= \langle q(x), p(x) \rangle. \end{aligned}$$

(ii) For all $p(x), q(x), r(x) \in P_2$ we have

$$\begin{aligned} \langle p(x) + q(x), r(x) \rangle &= (p_o + q_0) r_0 + (p_1 + q_1) r_1 + (p_2 + q_2) r_2 \\ &= (p_0 r_0 + p_1 r_1 + p_2 r_2) + (q_0 r_0 + q_1 r_1 + q_2 r_2) \\ &= \langle p(x), r(x) \rangle + \langle q(x), r(x) \rangle. \end{aligned}$$

(iii) For all $p(x), q(x) \in P_2$ and all $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \langle \lambda p(x), q(x) \rangle &= (\lambda p_0) q_0 + (\lambda p_1) q_1 + (\lambda p_2) q_2 \\ &= \lambda (p_0 q_0 + p_1 q_1 + p_2 q_2) \\ &= \lambda \langle p(x), q(x) \rangle. \end{aligned}$$

(iv) For all $p(x) \in P_2$ we have

$$\langle p(x), p(x) \rangle = p_0^2 + p_1^2 + p_2^2 \geq 0.$$

Moreover $\langle p(x), p(x) \rangle = 0$ if and only if $p_0 = p_1 = p_2 = 0$, i.e. $p(x)$ is the zero polynomial.

2.

$$\begin{aligned} \langle A, B \rangle = tr(B^T A) &= tr \left(\begin{pmatrix} 0 & -1 \\ 1 & -2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ -2 & 5 & 1 \end{pmatrix} \right) \\ &= tr \begin{pmatrix} 2 & -5 & -1 \\ 5 & -11 & 1 \\ -9 & 27 & 15 \end{pmatrix} = 2 - 11 + 15 = 6. \end{aligned}$$

$$
\begin{aligned}
\langle A, A\rangle = tr(A^T A) &= tr\left(\begin{pmatrix} 1 & -2 \\ -1 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ -2 & 5 & 1 \end{pmatrix}\right) \\
&= tr\begin{pmatrix} 5 & -11 & 1 \\ -11 & 26 & 2 \\ 1 & 2 & 10 \end{pmatrix} = 41.
\end{aligned}
$$

Thus $||A|| = \sqrt{\langle A, A\rangle} = \sqrt{41}$.

$$
\begin{aligned}
\langle B, B\rangle = tr(B^T B) &= tr\left(\begin{pmatrix} 0 & -1 \\ 1 & -2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ -1 & -2 & 6 \end{pmatrix}\right) \\
&= tr\begin{pmatrix} 1 & 2 & -6 \\ 2 & 5 & -9 \\ -6 & -9 & 45 \end{pmatrix} = 51.
\end{aligned}
$$

Thus $||B|| = \sqrt{\langle B, B, \rangle} = \sqrt{51}$.

3. Let us start with the inner product given in (d).

$$
\begin{aligned}
\langle x+3, x^2+x-1\rangle &= \int_{-1}^{1} (x+3)(x^2+x-1)dx \\
&= \int_{-1}^{1} (x^3+4x^2+2x-3)dx \\
&= \left[\frac{x^4}{4} + 4\frac{x^3}{3} + 2\frac{x^2}{2} - 3x\right]_{-1}^{1} \\
&= -\frac{10}{3}.
\end{aligned}
$$

$$
\begin{aligned}
\langle x+3, x+3\rangle &= \int_{-1}^{1} (x+3)^2 dx \\
&= \int_{-1}^{1} (x^2+6x+9)dx \\
&= \left[\frac{x^3}{3} + 6\frac{x^2}{2} + 9x\right]_{-1}^{1} \\
&= \frac{56}{3}.
\end{aligned}
$$

Thus we have $||x+3|| = \sqrt{\frac{56}{3}}$.

$$\begin{aligned}\langle x^2+x-1, x^2+x-1\rangle &= \int_{-1}^{1}(x^2+x-1)^2dx \\ &= \int_{-1}^{1}(x^4+2x^3-x^2-2x+1)dx \\ &= \left[\frac{x^5}{5}+2\frac{x^4}{4}-\frac{x^3}{3}-2\frac{x^2}{2}+x\right]_{-1}^{1} \\ &= \frac{26}{15}.\end{aligned}$$

Thus we have $||x^2+x-1|| = \sqrt{\frac{26}{15}}$.

Now using the inner product given in (e) we get

$$\langle x+3, x^2+x-1\rangle = 3.(-1)+1.1+0.1 = -2.$$

$$\langle x+3, x+3\rangle = 3.3+1.1+0.0 = 10,$$

so we have $||x+3|| = \sqrt{10}$.

$$\langle x^2+x-1, x^2+x-1\rangle = (-1).(-1)+1.1+1.1 = 3,$$

so we have $||x^2+x-1|| = \sqrt{3}$.

4. (a) This set is orthogonal as

$$\begin{aligned}&(1,1,1,1).(1,-1,0,0) = 1.1+1.(-1)+1.0+1.0 = 0,\\ &(1,1,1,1).(0,0,1,-1) = 1.0+1.0+1.1+1.(-1) = 0,\\ &(1,-1,0,0).(0,0,1,-1) = 1.0+(-1).0+0.1+0.(-1) = 0.\end{aligned}$$

But it is not orthonormal as for example

$$||(1,1,1,1)|| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2 \neq 1.$$

(b) This set is orthogonal as

$$\langle (1,0),(0,1)\rangle = 2.1.0+4.0.1 = 0,$$

but it is not orthonormal as for example

$$||(1,0)|| = \sqrt{\langle (1,0),(1,0)\rangle} = \sqrt{2.1.1+4.0.0} = \sqrt{2} \neq 1.$$

(c) This set is orthonormal (and thus also orthogonal) as

$$\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}\rangle = tr\left(\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = tr\begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix} = 0,$$

$$\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix}\rangle = tr\left(\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = tr\begin{pmatrix} 0 & \frac{-1}{\sqrt{6}} \\ 0 & 0 \end{pmatrix} = 0,$$

$$\langle \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix}\rangle = tr\begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{2}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{pmatrix} = 0,$$

thus these matrices are pairwise orthogonal. Moreover they all have norm 1 as shown below.

$$\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle = tr \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1,$$

thus $|| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} || = 1.$

$$\langle \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix} \rangle = tr \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = 1,$$

thus $|| \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix} || = 1.$

$$\langle \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix} \rangle = tr \begin{pmatrix} \frac{5}{6} & \frac{-2}{6} \\ \frac{-2}{6} & \frac{1}{6} \end{pmatrix} = 1,$$

thus $|| \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & 0 \end{pmatrix} || = 1.$

(d) This set is orthogonal as

$$\langle x, x^2 \rangle = \int_{-1}^{1} x^3 \, dx = 0,$$

$$\langle x, 1 - \frac{5}{3}x^2 \rangle = \int_{-1}^{1} (x - \frac{5}{3}x^3) \, dx = 0,$$

$$\langle x^2, 1 - \frac{5}{3}x^2 \rangle = \int_{-1}^{1} (x^2 - \frac{5}{3}x^4) dx = \frac{2}{3} - \frac{5}{3}.\frac{2}{5} = 0.$$

But it is not orthonormal as for instance

$$||x||^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \neq 1.$$

(e) This set is not orthogonal (or so it is not orthonormal) as for instance

$$\langle 1, 1 + x \rangle = 1.1 + 0.1 + 0.0 = 1 \neq 0.$$

5.

$$\begin{aligned} ||\mathbf{v_1} + \mathbf{v_2} + \ldots + \mathbf{v_r}||^2 &= \langle \mathbf{v_1} + \mathbf{v_2} + \ldots + \mathbf{v_r}, \mathbf{v_1} + \mathbf{v_2} + \ldots + \mathbf{v_r} \rangle \\ &= \langle \mathbf{v_1}, \mathbf{v_1} \rangle + \langle \mathbf{v_1}, \mathbf{v_2} \rangle + \ldots + \langle \mathbf{v_1}, \mathbf{v_r} \rangle \\ &\quad + \langle \mathbf{v_2}, \mathbf{v_1} \rangle + \langle \mathbf{v_2}, \mathbf{v_2} \rangle + \ldots + \langle \mathbf{v_2}, \mathbf{v_r} \rangle \\ &\quad + \ldots \\ &\quad + \langle \mathbf{v_r}, \mathbf{v_1} \rangle + \langle \mathbf{v_r}, \mathbf{v_2} \rangle + \ldots + \langle \mathbf{v_r}, \mathbf{v_r} \rangle \end{aligned}$$

using the definition of inner product $\langle , \rangle$ (i)(ii)(iii). But as the set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is orthogonal we have

$$\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0 \qquad \text{for } i \neq j.$$

Thus we get

$$\begin{aligned} ||\mathbf{v_1} + \mathbf{v_2} + \ldots + \mathbf{v_r}||^2 &= \langle \mathbf{v_1}, \mathbf{v_1} \rangle + \langle \mathbf{v_2}, \mathbf{v_2} \rangle + \ldots \langle \mathbf{v_r}, \mathbf{v_r} \rangle \\ &= ||\mathbf{v_1}||^2 + ||\mathbf{v_2}||^2 + \ldots + ||\mathbf{v_r}||^2 \end{aligned}$$

as required.