Linear Algebra: Solutions to Exercise Sheet 7

1. (a) Set $\mathbf{v_1} = (1,0,0)$ as it already has norm 1. Now $\mathbf{w_2} = (1,1,0) - (1,0,0) = (0,1,0) = \mathbf{v_2}$ as it already has norm 1. And finally $\mathbf{w_3} = (1,1,1) - (1,0,0) - (0,1,0) = (0,0,1) = \mathbf{v_3}$. Thus in this case we obtain the standard orthonormal basis.

   (b) We start with $\{\mathbf{u_1} = (2,0,3), \mathbf{u_2} = (-1,0,5), \mathbf{u_3} = (10,-7,2)\}$.
   As $||(2,0,3)||^2 = 4+9 = 13$ we set $\mathbf{v_1} = \frac{1}{\sqrt{13}}(2,0,3) = (\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}})$.
   Now $\mathbf{w_2} = (-1,0,5) - \frac{13}{\sqrt{13}}(\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}) = (-1,0,5) - (2,0,3) = (-3,0,2)$.
   As $||\mathbf{w_2}||^2 = 13$ we set $\mathbf{v_2} = \frac{1}{\sqrt{13}}(-3,0,2) = (\frac{-3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}})$.
   Finally, $\mathbf{w_3} = (10,-7,2) - \frac{26}{\sqrt{13}}(\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}) - (\frac{-26}{\sqrt{13}})(\frac{-3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}) = (10,-7,2) - 2(2,0,3) + 2(-3,0,2) = (0,-7,0)$. As $||\mathbf{w_3}||^2 = 49$ we set $\mathbf{v_3} = \frac{1}{7}(0,-7,0) = (0,-1,0)$.
   So we get the orthonormal basis $\{(\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}), (\frac{-3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}), (0,-1,0)\}$.

2. Let $u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
   Now we have
$$v_1 = \frac{1}{||u_1||}u_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$
$$\begin{aligned} w_2 &= u_2 - \langle u_2, v_1\rangle v_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-3}{4} \end{pmatrix}. \end{aligned}$$
   As $||w_2|| = \frac{\sqrt{3}}{2}$ we have $v_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{-3}{2\sqrt{3}} \end{pmatrix}$.
$$\begin{aligned} w_3 &= u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \frac{1}{\sqrt{3}}\begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{-3}{2\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{-3}{6} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & 0 \end{pmatrix}. \end{aligned}$$
   As $||w_3|| = \frac{\sqrt{6}}{3}$ we have $v_3 = \frac{3}{\sqrt{6}}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-2}{3} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} & 0 \end{pmatrix}$.
$$\begin{aligned} w_4 &= u_4 - \langle u_4, v_1\rangle v_1 - \langle u_4, v_2\rangle v_2 - \langle u_4, v_3\rangle v_3 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \frac{1}{2\sqrt{3}}\begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{-3}{2\sqrt{3}} \end{pmatrix} - \frac{1}{\sqrt{6}}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{-3}{12} \end{pmatrix} - \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{-2}{6} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

As $||w_4|| = \frac{1}{\sqrt{2}}$ we have $v_4 = \sqrt{2}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 0 & 0 \end{pmatrix}$.

3. Let $u_1 = 2 + 3x^2$, $u_2 = 5x^2 - 1$ and $u_3 = 10 - 7x + 2x^2$.
Now $v_1 = \frac{1}{||u_1||}u_1 = \frac{1}{\sqrt{13}}(2 + 3x^2)$.

$$\begin{aligned} w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 \\ &= 5x^2 - 1 - (2 + 3x^2) = 2x^2 - 3. \end{aligned}$$

As $||w_2|| = \sqrt{13}$ we have $v_2 = \frac{1}{\sqrt{13}}(2x^2 - 3)$.

$$\begin{aligned} w_3 &= u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \\ &= 10 - 7x + 2x^2 - 2(2 + 3x^2) + 2(2x^2 - 3) = -7x. \end{aligned}$$

As $||w_3|| = 7$ we have $v_3 = -x$.

4. We start with the basis of P_2 given by $\{\mathbf{u_1} = 1, \mathbf{u_2} = x, \mathbf{u_3} = x^2\}$ and use the Gram-Schmidt process to construct an orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

Set $\mathbf{v_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||}$. As

$$||1||^2 = \int_{-1}^{1} 1dx = [x]_{-1}^{1} = 2,$$

we have

$$\mathbf{v_1} = \frac{1}{\sqrt{2}}.$$

Set $\mathbf{w_2} = \mathbf{u_2} - \langle \mathbf{u_2}, \mathbf{v_1} \rangle \mathbf{v_1}$. As

$$\langle \mathbf{u_2}, \mathbf{v_1} \rangle = \int_{-1}^{1} \frac{x}{\sqrt{2}} dx = \left[\frac{x^2}{2\sqrt{2}}\right]_{-1}^{1} = 0,$$

we have

$$\mathbf{w_2} = \mathbf{u_2} = x.$$

Now we set $\mathbf{v_2} = \frac{\mathbf{w_2}}{||\mathbf{w_2}||}$. As

$$||\mathbf{w_2}||^2 = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

we have

$$\mathbf{v_2} = \frac{\sqrt{3}}{\sqrt{2}}x.$$

Set $\mathbf{w_3} = \mathbf{u_3} - \langle \mathbf{u_3}, \mathbf{v_1} \rangle \mathbf{v_1} - \langle \mathbf{u_3}, \mathbf{v_2} \rangle \mathbf{v_2}$. As

$$\langle \mathbf{u_3}, \mathbf{v_1} \rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx = \left[\frac{x^3}{3\sqrt{2}}\right]_{-1}^{1} = \frac{2}{3\sqrt{2}}$$

and

$$\langle \mathbf{u_3}, \mathbf{v_2} \rangle = \int_{-1}^{1} x^2 \frac{\sqrt{3}}{\sqrt{2}} x dx = \left[\frac{\sqrt{3}}{\sqrt{2}} \frac{x^4}{4}\right]_{-1}^{1} = 0$$

we have

$$\mathbf{w_3} = x^2 - \frac{2}{3\sqrt{2}}\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}.$$

Now we set $\mathbf{v_3} = \frac{\mathbf{w_3}}{||\mathbf{w_3}||}$. As

$$||\mathbf{w_3}||^2 = \int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \left[\frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{1}{9}x\right]_{-1}^{1} = \frac{8}{45}$$

we have

$$\mathbf{v_3} = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}).$$

Thus we have constructed an orthonormal basis for P_2 given by

$$\{\mathbf{v_1} = \frac{1}{\sqrt{2}},\ \mathbf{v_2} = \frac{\sqrt{3}}{\sqrt{2}}x,\ \mathbf{v_3} = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})\}$$

We have

$$x^2-2x+3 = \langle x^2-2x+3, \frac{1}{\sqrt{2}}\rangle\frac{1}{\sqrt{2}} + \langle x^2-2x+3, \frac{\sqrt{3}}{\sqrt{2}}x\rangle\frac{\sqrt{3}}{\sqrt{2}}x + \langle x^2-2x+3, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2-\frac{1}{3})\rangle\frac{3\sqrt{5}}{2\sqrt{2}}(x^2-\frac{1}{3}).$$

As

$$\begin{aligned}
\langle x^2 - 2x + 3, \frac{1}{\sqrt{2}}\rangle &= \int_{-1}^{1}(x^2 - 2x + 3)\frac{1}{\sqrt{2}}dx = \frac{20}{3\sqrt{2}} \\
\langle x^2 - 2x + 3, \frac{\sqrt{3}}{\sqrt{2}}x\rangle &= \int_{-1}^{1}(x^2 - 2x + 3)\frac{\sqrt{3}}{\sqrt{2}}x dx = -\frac{4\sqrt{3}}{3\sqrt{2}} \\
\langle x^2 - 2x + 3, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})\rangle &= \int_{-1}^{1}(x^2 - 2x + 3)\frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})dx = \frac{4\sqrt{5}}{15\sqrt{2}}
\end{aligned}$$

we have $x^2 - 2x + 3 = \frac{20}{3\sqrt{2}}(\frac{1}{\sqrt{2}}) - \frac{4\sqrt{3}}{3\sqrt{2}}(\frac{\sqrt{3}}{\sqrt{2}}x) + \frac{4\sqrt{5}}{15\sqrt{2}}(\frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}))$.

5. Note that if we write vectors in $\mathbb{R}^n$ as column vectors we can write the dot product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as the product of two matrices (one with just one row and the other with just one column) $\mathbf{x}\cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$.

   (a) Write $\mathbf{c}(i)$ for the i-th column of the matrix A. Then using the definition of matrix multiplication we see that $A^TA = I$ is equivalent to saying that

   $$\mathbf{c}(i)^T\mathbf{c}(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

   In other words, the columns of A form an orthonormal set. Note that $A^TA = I$ if and only if $AA^T = I$, thus the same result is true for the rows of the matrix A.

   (b) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

   $$\begin{aligned}
   (A\mathbf{x})\cdot(A\mathbf{y}) &= (A\mathbf{x})^T(A\mathbf{y}) \\
   &= (\mathbf{x}^TA^T)(A\mathbf{y}) \qquad \text{using } (BC)^T = C^TB^T \\
   &= \mathbf{x}^T(A^TA)\mathbf{y} \\
   &= \mathbf{x}^T\mathbf{y} \qquad \text{using } A^TA = I \text{ as } A \text{ is orthogonal} \\
   &= \mathbf{x}\cdot\mathbf{y}.
   \end{aligned}$$

(c) Using (b) we get for all $\mathbf{x} \in \mathbb{R}^n$ that

$$||A\mathbf{x}||^2 = (A\mathbf{x})\cdot (A\mathbf{x}) = \mathbf{x}\cdot \mathbf{x} = ||\mathbf{x}||^2,$$

thus $||A\mathbf{x}|| = ||\mathbf{x}||$.

Recall that the dot product can be defined as $\mathbf{x}\cdot \mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}||\cos\theta$ where θ is the angle between the vector $\mathbf{x}$ and the vector $\mathbf{y}$. If we view the matrix A as a linear map from $\mathbb{R}^n$ to itself then (b) and (c) tell us that the map A preserves distances and angles between vectors.