Linear Algebra Exam January 2007: Solutions

1. (a) i. U is not a subspace of $M(2,2)$.
Check that condition (S2) fails by taking $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in U$, but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin U$

ii. V is a subspace of P_n.
(S1) the zero polynomial is in V.
(S2) if $p, q \in V$, i.e. $p(3) = q(3) = 0$, then $p + q \in V$ as $(p+q)(3) = p(3) + q(3) = 0$.
(S3) If $p \in V$ and $\lambda \in \mathbb{R}$ then $\lambda p \in V$ as $(\lambda p)(3) = \lambda(p(3)) = \lambda 0 = 0$.

iii. W is a subspace of $\mathbb{R}^4$.
(S1) $(0,0,0,0) \in W$ as $0 + 0 = 0$.
(S2) If $(x,y,z,t) \in W$ and $(x',y',z',t') \in W$ then

$$(x,y,z,t) + (x',y',z',t') = (x+x', y+y', z+z', t+t') \in W$$

as $(y+y') = (z+t) + (z'+t') = (z+z') + (t+t')$.
(S3) If $(x,y,z,t) \in W$ and $\lambda \in \mathbb{R}$ then

$$\lambda(x,y,z,t) = (\lambda x, \lambda y, \lambda z, \lambda t) \in W$$

as $\lambda y = \lambda(z+t) = \lambda z + \lambda t$.

[8]

(b) Take for example $\{(1,0,0), (0,1,0), (3,5,-4)\}$. It is easy to check that these vectors are linearly independent, and as $\dim \mathbb{R}^3 = 3$ it must be a basis for $\mathbb{R}^3$.

[4]

(c) i. Is this set linearly independent? No, as the $\dim \mathbb{R}^3 = 3$ so any linearly independent set has at most three vectors. Thus it is not a basis either.
Is it a spanning set? yes. Need to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$(a,b,c) = \lambda_1(1,0,1) + \lambda_2(1,1,0) + \lambda_3(0,1,1) + \lambda_4(1,1,1).$$

Take for example $\lambda_1 = \frac{1}{2}(c-b+a)$, $\lambda_2 = \frac{1}{2}(b-c+a)$, $\lambda_3 = \frac{1}{2}(b+c-a)$ and $\lambda_4 = 0$.

ii. Is it linearly independent? Write

$$\lambda_1 5 + \lambda_2(2 + x - 3x^2) + \lambda_3(4x - 1) = 0$$

$$(5\lambda_1 + 2\lambda_2 - \lambda_3) + (\lambda_2 + 4\lambda_3)x - 3\lambda_2 x^2 = 0$$

This implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$, thus this set is linearly independent. As $\dim P_2 = 3$ and we have three linearly independent vectors, it is automatically a basis for P_2 (and hence is also spanning).

[8]

2. (a) i. f is linear as for all $(a,b),(a',b') \in \mathbb{R}^2$ we have

$$\begin{aligned} f((a,b)+(a',b')) &= f(a+a', b+b') \\ &= (a+a'+b+b')x^5 \\ &= (a+b)x^5 + (a'+b')x^5 \\ &= f(a,b) + f(a',b') \end{aligned}$$

and for all $(a,b) \in \mathbb{R}^2$ and all $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} f(\lambda(a,b)) &= f(\lambda a, \lambda b) \\ &= (\lambda a + \lambda b)x^5 \\ &= \lambda(a+b)x^5 \\ &= \lambda f(a,b). \end{aligned}$$

ii. f is not linear. Take for example $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M(2,2)$ and $\lambda = 2$ then

$$f(2A) = (4,4) \neq 2f(A) = (2,2).$$

[6]

(b) Let $f : V \to W$ be a linear map from a vector space V to a vector space W. The image of f is defined by $\operatorname{Im} f = \{w \in W \,:\, w = f(v) \text{ for some } v \in V\}$. The kernel of f is defined by $\operatorname{Ker} f = \{v \in V \,:\, f(v) = 0\}$. The rank of f is the dimension of the image of f. The nullity of f is the dimension of the kernel of f.

The rank-nullity theorem says

$$\dim V = \operatorname{rank} f + \operatorname{nullity} f.$$

[4]

(c) $\operatorname{nullity} f = \dim \operatorname{Ker} f = 1$, thus $\operatorname{Ker} f \neq \{\mathbf{0}\}$ and so f is not injective. Now using the rank-nullity theorem we have

$$\dim P_2 = \operatorname{rank} f + 1$$

and as $\dim P_2 = 3$ we see that $\operatorname{rank} f = \dim \operatorname{Im} f = 2$. This implies that $\operatorname{Im} f$ is a proper subspace of $\mathbb{R}^3$ and hence f is not surjective.

[4]

(d)

$$\begin{aligned} \operatorname{Ker} f &= \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y = z+t = 0\} \\ &= \{(x,-x,z,-z) \mid x,z \in \mathbb{R}\}. \end{aligned}$$

Thus f is not injective. A basis for $\operatorname{Ker} f$ is given by $\{(1,-1,0,0),(0,0,1,-1)\}$. It is a spanning set as

$$(x,-x,z,-z) = x(1,-1,0,0) + z(0,0,1,-1)$$

for all $x, z \in \mathbb{R}$, and these two vectors are clearly linearly independent.
Now using the Rank-Nullity theorem we have

$$\dim \mathbb{R}^4 = \operatorname{rank} f + 2$$

and as $\dim \mathbb{R}^4 = 4$ we have $\operatorname{rank} f = 2$. This implies that $\operatorname{Im} f$ is a proper subspace of $\mathbb{R}^3$ and so f is not surjective. Now as $\dim \operatorname{Im} f = 2$,

$$f(1,0,0,0) = (1,0,0) \in \operatorname{Im} f$$

$$f(0,0,1,0) = (0,0,1) \in \operatorname{Im} f$$

and $(1,0,0)$ and $(0,0,1)$ and linearly independent, $\{(1,0,0),(0,0,1)\}$ form a basis for $\operatorname{Im} f$.

[6]

3. (a) An eigenvector for an $n \times n$ matrix A is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$. An eigenvalue for A is a real number λ such that there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$.

[3]

(b)

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue 3.

[3]

(c)

$$\det \begin{pmatrix} -\lambda & 2 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 2 & -\lambda \end{pmatrix} = 0$$

This gives

$$-\lambda^3 + \lambda^2 + 5\lambda + 3 = -(\lambda+1)^2(\lambda-3) = 0.$$

Thus the eigenvalues of A are 3 and -1.

[3]

When $\lambda = 3$ we have

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(3) = \{(x, x, x) \, : \, x \in \mathbb{R}\}$$

with basis given by $\{(1, 1, 1)\}$ (clearly spanning and linearly independent).

[3]

When $\lambda = -1$ we have

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(-1) = \{(x, y, -x - 2y) \, : \, x, y \in \mathbb{R}\}$$

with basis given by $\{(1, 0, -1), (0, 1, -2)\}$ (clearly spanning and linearly independent).

[3]

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

$$P^{-1}AP = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

[5]

4. (a) The norm of a matrix A is given by $||A|| = \sqrt{tr(A^T A)}$.
$|| \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix} || = \sqrt{39}.$

[2]

(b) Two matrices A and B are orthogonal if and only if $\langle A, B\rangle = tr(B^T A) = 0$. The two matrices $\begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ are not orthogonal as

$$\langle \begin{pmatrix} 6 & 3 \\ 3 & 27 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \rangle = 27 \neq 0.$$

[2]

(c) A set of matrices $\{B_1, \ldots, B_k\}$ in $M(2,2)$ is orthonormal if $\langle B_i, B_j\rangle = 0$ for all $i \neq j$ and $||B_i|| = 1$ for all $i = 1, \ldots k$. This set is orthonormal as

$$\langle A_1, A_2\rangle = tr\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0,$$
$$\langle A_1, A_3\rangle = tr\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0,$$
$$\langle A_2, A_3\rangle = tr\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0$$

and

$$||A_1||^2 = tr\begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix} = 1,$$
$$||A_2||^2 = tr\begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix} = 1,$$
$$||A_3||^2 = tr\begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix} = 1.$$

[6]

(d) First define

$$A_4' = B - \langle B, A_1\rangle A_1 - \langle B, A_2\rangle A_2 - \langle B, A_3\rangle A_3.$$

As

$$\langle B, A_1\rangle = tr\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0,$$
$$\langle B, A_2\rangle = tr\begin{pmatrix} 0 & * \\ * & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}},$$
$$\langle B, A_3\rangle = tr\begin{pmatrix} -1 & * \\ * & 0 \end{pmatrix} = -1,$$

we have

$$A_4' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Now $||A_4'||^2 = tr\begin{pmatrix} 0 & * \\ * & \frac{1}{2} \end{pmatrix} = \frac{1}{2}$, so $||A_4|| = \frac{1}{\sqrt{2}}$. Now we take

$$A_4 = \sqrt{2}A_4' = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

[6]

(e) Let $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\begin{aligned}
\lambda_1 &= \langle C, A_1 \rangle = tr(A_1^T C) = tr(A_1) = 1 \\
\lambda_2 &= \langle C, A_2 \rangle = tr(A_2^T C) = tr(A_2) = \frac{1}{\sqrt{2}} \\
\lambda_3 &= \langle C, A_3 \rangle = tr(A_3^T C) = tr(A_3) = 0 \\
\lambda_4 &= \langle C, A_4 \rangle = tr(A_4^T C) = tr(A_4) = -\frac{\sqrt{2}}{2}.
\end{aligned}$$

[4]