Linear Algebra Exam May 2006: Solutions

1. (a) i. U is a subspace of $M(2,2)$.

(S1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ as $0+0+0+0=0$.

(S2) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} \in U$$

as $(a+a')+(b+b')+(c+c')+(d+d') = (a+b+c+d)+(a'+b'+c'+d') = 0+0 = 0$.

(S3) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ and $\lambda \in \mathbb{R}$ then

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \in U$$

as $\lambda a + \lambda b + \lambda c + \lambda d = \lambda(a+b+c+d) = \lambda 0 = 0$.

ii. V is not a subspace of $\mathbb{R}^3$.

(S3) fails. Take $(1,0,1) \in V$ then $2(1,0,1) = (2,0,2) \notin V$.

iii. W is a subspace of P_2.

(S1) The zero polynomial $0 + 0x + 0x^2 \in W$.

(S2) If $a_0 + a_1x + a_2x^2 \in W$ and $b_0 + b_1x + b_2x^2 \in W$ then

$$(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2 \in W$$

as $a_0 + b_0 = (a_1 + 2a_2) + (b_1 + 2b_2) = (a_1 + b_1) + 2(a_2 + b_2)$.

(S3) If $a_0 + a_1x + a_2x^2 \in W$ and $\lambda \in \mathbb{R}$ then

$$\lambda(a_0 + a_1x + a_2x^2) = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 \in W$$

as $\lambda a_0 = \lambda(a_1 + 2a_2) = \lambda a_1 + 2\lambda a_2$.

[8]

(b) Take $\{1, x, x^2, \ldots, x^n\}$. The dimension of P_n is equal to $n+1$.

[4]

(c) i. Is this set lineraly independent?

$$\lambda_1(1,5,-2) + \lambda_2(-2,1,1) + \lambda_3(0,0,3) = (0,0,0)$$

$$\begin{aligned} \lambda_1 - 2\lambda_2 &= 0 \\ 5\lambda_1 + \lambda_2 &= 0 \\ -2\lambda_1 + \lambda_2 + 3\lambda_3 &= 0 \end{aligned}$$

The only solution to this system of linear equation is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus these vectors are linearly independent. As we have 3 linearly independent vectors in the 3-dimensional space $\mathbb{R}^3$, these vectors form a basis for $\mathbb{R}^3$. In particular, they form a spanning set for $\mathbb{R}^3$.

ii. This set is not linearly independent as it contains 4 vectors and $\dim \mathbb{R}^3 = 3$. So it is not a basis.
Is it a spanning set?

$$(x, y, z) = \lambda_1(1, 0, 0) + \lambda_2(2, 1, 0) + \lambda_3(3, 2, 1) + \lambda_4(-1, -1, -1)$$

As $(-1, -1, -1)$ can be written as a linear combination of the other three vectors we can set $\lambda_4 = 0$ and we get

$$\lambda_3 = z, \quad \lambda_2 = y - 2z \quad \text{and } \lambda_1 = x - 2y + z.$$

[8]

2. (a) To check that f is linear we need to check the following two conditions.

$$f((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)) = f((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2)$$
$$= ((a_1 + b_1) + (a_0 + b_0), (a_2 + b_2) + (a_0 + b_0), a_1 + b_1, a_2 + b_2)$$
$$= (a_1 + a_0, a_2 + a_0, a_1, a_2) + (b_1 + b_0, b_2 + b_0, b_1, b_2)$$
$$= f(a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2).$$

$$\begin{aligned} f(\lambda(a_0 + a_1x + a_2x^2)) &= f(\lambda a_0 + \lambda a_1 x + \lambda a_2 x^2) \\ &= (\lambda a_1 + \lambda a_0, \lambda a_2 + \lambda a_0, \lambda a_1, \lambda a_2) \\ &= \lambda(a_1 + a_0, a_2 + a_0, a_1, a_2) \\ &= \lambda f(a_0 + a_1x + a_2x^2). \end{aligned}$$

Now to find the matrix representing f in those basis we consider

$$f(1) = (1, 1, 0, 0) = 1\mathbf{e_1} + 1\mathbf{e_2} + 0\mathbf{e_3} + 0\mathbf{e_4}$$
$$f(x) = (1, 0, 1, 0) = 1\mathbf{e_1} + 0\mathbf{e_2} + 1\mathbf{e_3} + 0\mathbf{e_4}$$
$$f(x^2) = (0, 1, 0, 1) = 0\mathbf{e_1} + 1\mathbf{e_2} + 0\mathbf{e_3} + 1\mathbf{e_4}$$

and so the matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[6]

(b) No, there is no such linear map as $f(2,0,0) = f(2(1,0,0)) \neq 2f(1,0,0)$.

[3]

(c) Let $f : V \to W$ be a linear map from a vector space V to a vector space W. The image of f is defined by $\mathrm{Im}\, f = \{w \in W \,:\, w = f(v) \text{ for some } v \in V\}$. The kernel of f is defined by $\mathrm{Ker}\, f = \{v \in V \,:\, f(v) = 0\}$. The rank of f is the dimension of the image of f. The nullity of f is the dimension of the kernel of f.

The rank-nullity theorem says

$$\dim V = \mathrm{rank}\, f + \mathrm{nullity}\, f.$$

[3]

(d) f is not injective as $f(2,0,0) = (2,0,0) = f(1,1,0)$.
f is not surjective as $(0,1,0) \notin \mathrm{Im}\, f$.

$$\begin{aligned} \mathrm{Ker}\, f &= \{(x,y,z) \in \mathbb{R}^3 \,:\, (x+y+z, z, z) = (0,0,0)\} \\ &= \{(x,-x,0) \,:\, x \in \mathbb{R}\}. \end{aligned}$$

We claim that $\{(1,-1,0)\}$ is a basis for $\mathrm{Ker}\, f$. It contains one non-zero vector so it is certainly linearly independent and as

$$(x,-x,0) = x(1,-1,0) \qquad \forall x \in \mathbb{R}$$

it is a spanning set as well, thus it is a basis.

Using the rank-nullity theorem we see that $\dim \mathrm{Im}\, f = 3 - 1 = 2$. As $(1,0,0) = f(1,0,0)$ and $(1,1,1) = f(0,0,1)$ are in the image of f and they are linearly independent, they form a basis for the image of f.

[8]

3. (a) An eigenvector for A is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$. An eigenvalue for A is a real number λ such that there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$.

[3]

(b) Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors then there is an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal.

[3]

(c)

$$\det \begin{pmatrix} 7-\lambda & 4 & -8 \\ 8 & 3-\lambda & -8 \\ 8 & 4 & -9-\lambda \end{pmatrix} = 0$$

This gives

$$(7-\lambda)[(3-\lambda)(-9-\lambda)+32]-4[8(-9-\lambda)+64]-8[32-8(3-\lambda)]=0$$

and so

$$-\lambda^3+\lambda^2+5\lambda+3=-(\lambda+1)^2(\lambda-3)=0.$$

Thus the eigenvalues of A are -1 and 3.

[3]

When $\lambda=-1$ we have

$$\begin{pmatrix} 8 & 4 & -8 \\ 8 & 4 & -8 \\ 8 & 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(-1)=\{(x,-2x+2z,z)\ :\ x,z\in\mathbb{R}\}$$

with basis given by $\{(1,-2,0),(0,2,1)\}$ (clearly spanning and linearly independent).

[3]

When $\lambda=3$ we have

$$\begin{pmatrix} 4 & 4 & -8 \\ 8 & 0 & -8 \\ 8 & 4 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(3)=\{(x,x,x)\ :\ x\in\mathbb{R}\}$$

with basis given by $\{(1,1,1)\}$ (clearly spanning and linearly independent).

[3]

$$P=\begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix},\qquad P^{-1}=\begin{pmatrix} -1 & -1 & 2 \\ -2 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}.$$

$$P^{-1}AP=\begin{pmatrix} -1 & -1 & 2 \\ -2 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} 7 & 4 & -8 \\ 8 & 3 & -8 \\ 8 & 4 & -9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

[5]

4. (a) The norm of $\mathbf{x}$ is given by $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.
$||(-1, 2, -5, 3)|| = \sqrt{39}$.

[2]

(b) Two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. The two vectors $(1, 3, 0, 2)$ and $(-2, 0, 7, -1)$ are not orthogonal as $\langle (1, 3, 0, 2), (-2, 0, 7, -1) \rangle = -2 - 2 = -4 \neq 0$.

[2]

(c) A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ in $\mathbb{R}^4$ is orthonormal if $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0$ for all $i \neq j$ and $||\mathbf{v_i}|| = 1$ for all $i = 1, \ldots k$. This set is orthonormal as

$$\langle (1,0,0,0), (0, \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}) \rangle = 0,$$
$$\langle (1,0,0,0), (0, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}) \rangle = 0,$$
$$\langle (0, \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}), (0, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}) \rangle = 0,$$

and

$$||(1,0,0,0)|| = ||(0, \frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}})|| = ||(0, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}})|| = 1.$$

[6]

(d) Write $\mathbf{u_1} = (0, 0, 0, 1)$, $\mathbf{u_2} = (1, 0, 1, 1)$, $\mathbf{u_3} = (1, 1, 1, 0)$ and $\mathbf{u_4} = (0, -1, 1, 0)$. We construct an orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. First let $\mathbf{v_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||} = (0, 0, 0, 1)$. Next, set

$$\begin{aligned} \mathbf{w_2} &= \mathbf{u_2} - \langle \mathbf{u_2}, \mathbf{v_1} \rangle \mathbf{v_1} \\ &= (1,0,1,1) - (0,0,0,1) = (1,0,1,0) \end{aligned}$$

and

$$\mathbf{v_2} = \frac{\mathbf{w_2}}{||\mathbf{w_2}||} = \frac{1}{\sqrt{2}}(1,0,1,0).$$

Next, set

$$\begin{aligned} \mathbf{w_3} &= \mathbf{u_3} - \langle \mathbf{u_3}, \mathbf{v_1} \rangle \mathbf{v_1} - \langle \mathbf{u_3}, \mathbf{v_2} \rangle \mathbf{v_2} \\ &= (1,1,1,0) - 0(0,0,0,1) - (1,0,1,0) = (0,1,0,0) \end{aligned}$$

and

$$\mathbf{v_3} = \frac{\mathbf{w_3}}{||\mathbf{w_3}||} = (0,1,0,0).$$

Finally, set

$$\begin{aligned} \mathbf{w_4} &= \mathbf{u_4} - \langle \mathbf{u_4}, \mathbf{v_1} \rangle \mathbf{v_1} - \langle \mathbf{u_4}, \mathbf{v_2} \rangle \mathbf{v_2} - \langle \mathbf{u_4}, \mathbf{v_3} \rangle \mathbf{v_3} \\ &= (0,-1,1,0) - 0(0,0,0,1) - \frac{1}{2}(1,0,1,0) + (0,1,0,0) = (-\frac{1}{2}, 0, \frac{1}{2}, 0) \end{aligned}$$

and

$$\mathbf{v_4} = \frac{\mathbf{w_4}}{||\mathbf{w_4}||} = \frac{1}{\sqrt{2}}(-1, 0, 1, 0).$$

[10]