Linear Algebra Exam May 2007: Solutions

1. (a) i. U is a subspace of $M(2,2)$. Need to check that conditions (S1)-(S3) are satisfied.
(S1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ as $0+0=0+0$.
(S2) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U$ (i.e. $a+d=b+c$ and $a'+d'=b'+c'$) then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} \in U$$
as
$$\begin{aligned}(a+a')+(d+d') &= (a+d)+(a'+d') \\ &= (b+c)+(b'+c') \\ &= (b+b')+(c+c').\end{aligned}$$
(S3) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ and $\lambda \in \mathbb{R}$ then
$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \in U$$
as
$$\lambda a + \lambda d = \lambda(a+d) = \lambda(b+c) = \lambda b + \lambda c.$$

ii. V is a subspace of P_3.
(S1) the zero polynomial is in V.
(S2) if $p, q \in V$, i.e. $p(-2) = q(-2) = 0$, then $p+q \in V$ as $(p+q)(-2) = p(-2)+q(-2) = 0$.
(S3) If $p \in V$ and $\lambda \in \mathbb{R}$ then $\lambda p \in V$ as $(\lambda p)(-2) = \lambda(p(-2)) = \lambda 0 = 0$.

iii. W is not a subspace of $\mathbb{R}^n$. Check that (S3) fails. Take $(2,1,1,\ldots,1) \in W$ and $\lambda = -1 \in \mathbb{R}$ then $(-1)(2,1,1,\ldots,1) = (-2,-1,-1,\ldots,-1) \notin W$ as $-2 < -1$.

[8]

(b) Take for example $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. As it has 4 elements, $\dim M(2,2) = 4$.

[4]

(c) i. Is this set linearly independent? No. As the $\dim \mathbb{R}^3 = 3$ any linearly independent set has at most three vectors. Thus it is not a basis either.
Is it a spanning set? yes. Need to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that
$$(a,b,c) = \lambda_1(1,0,2) + \lambda_2(1,2,0) + \lambda_3(0,1,2) + \lambda_4(2,2,2).$$
Take for example $\lambda_1 = \frac{1}{6}(4a-2b+c)$, $\lambda_2 = \frac{1}{6}(2a+2b-c)$, $\lambda_3 = \frac{1}{3}(-2a+b+c)$ and $\lambda_4 = 0$.

ii. Is it linearly independent? Write

$$\lambda_1(5,0,0) + \lambda_2(2,-1,-3) + \lambda_3(-1,4,0) = (0,0,0)$$

This is equivalent to the system of equations

$$\begin{cases} 5\lambda_1 + 2\lambda_2 - \lambda_3 = 0 \\ \lambda_2 + 4\lambda_3 = 0 \\ -3\lambda_2 = 0 \end{cases}$$

The only solution is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus this set is linearly independent. As $\dim \mathbb{R}^3 = 3$ and we have three linearly independent vectors, it is automatically a basis for $\mathbb{R}^3$ (and hence is also spanning).

[8]

2. (a) A map $f : V \to W$ is linear if and only if the following two conditions are satisfied:

$$(i)\ f(u+v) = f(u) + f(v) \qquad \forall u, v \in V$$

$$(ii)\ f(\lambda v) = \lambda f(v) \qquad \forall v \in V, \lambda \in \mathbb{R}$$

[2]

(b) No. If the map were linear then we would have

$$\begin{aligned} f(2,1) &= f((1,0)+(1,1)) \\ &= f(1,0) + f(1,1) \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}. \end{aligned}$$

[4]

(c) $(x,y) = (x-y)(1,0) + y(1,1)$ so we must have

$$\begin{aligned} f(x,y) &= (x-y)f(1,0) + yf(1,1) \\ &= (x-y)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} x-y & x-y \\ y & y \end{pmatrix}. \end{aligned}$$

[4]

(d) Let $f : V \to W$ be a linear map from a vector space V to a vector space W. The image of f is defined by $\mathrm{Im}\, f = \{w \in W \,:\, w = f(v) \text{ for some } v \in V\}$. The kernel of f is defined by $\mathrm{Ker}\, f = \{v \in V \,:\, f(v) = 0\}$. The rank of f is the dimension of the image of f. The nullity of f is the dimension of the kernel of f.

The rank-nullity theorem says that if V is finite dimensional then we have

$$\dim V = \mathrm{rank}\, f + \mathrm{nullity}\, f.$$

[4]

(e)

$$\begin{aligned} \mathrm{Ker}\, f &= \{(x, y, z) \in \mathbb{R}^3 \mid x + y = y + z = 0\} \\ &= \{(-y, y, -y) \mid y \in \mathbb{R}\}. \end{aligned}$$

As $\mathrm{Ker}\, f \neq 0$ we know that f is not injective. A basis for $\mathrm{Ker}\, f$ is given by $\{(-1, 1, -1)\}$ (clearly spanning and linearly independent).
Now using the Rank-Nullity theorem we have

$$\dim \mathbb{R}^3 = \mathrm{rank}\, f + 1$$

and as $\dim \mathbb{R}^3 = 3$ we have $\mathrm{rank}\, f = 2$. This implies that $\mathrm{Im}\, f$ is a proper subspace of $\mathbb{R}^4$ and so f is not surjective. Now as $\dim \mathrm{Im}\, f = 2$,

$$f(1, 0, 0) = (1, 0, 1, 0) \in \mathrm{Im}\, f$$

$$f(0, 0, 1) = (0, 1, 0, 1) \in \mathrm{Im}\, f$$

and $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ are linearly independent, $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$ form a basis for $\mathrm{Im}\, f$.

[6]

3. (a) An eigenvector for an $n \times n$ matrix A is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$. An eigenvalue for A is a real number λ such that there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$.

[3]

(b) Suppose A has n linearly independent eigenvectors then there exists an invertible matrix P (whose columns are these eigenvectors) such that $P^{-1}AP$ is diagonal.

[3]

(c)

$$\det \begin{pmatrix} -4-\lambda & 0 & -2 \\ -4 & -2-\lambda & -4 \\ 4 & 0 & 2-\lambda \end{pmatrix} = 0$$

This gives

$$-\lambda^3 - 4\lambda^2 - 4\lambda = -\lambda(\lambda + 2)^2 = 0.$$

Thus the eigenvalues of A are 0 and -2.

[3]

When $\lambda = 0$ we have

$$\begin{pmatrix} -4 & 0 & -2 \\ -4 & -2 & -4 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(3) = \{(x, 2x, -2x) \, : \, x \in \mathbb{R}\}$$

with basis given by $\{(1, 2, -2)\}$ (clearly spanning and linearly independent).

[3]

When $\lambda = -2$ we have

$$\begin{pmatrix} -2 & 0 & -2 \\ -4 & 0 & -4 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the eigenspace is given by

$$s_A(-1) = \{(x, y, -x) \, : \, x, y \in \mathbb{R}\}$$

with basis given by $\{(1, 0, -1), (0, 1, 0)\}$ (clearly spanning and linearly independent).

[3]

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -4 & 0 & -2 \\ -4 & -2 & -4 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

[5]

4. (a) (i) For all $p(x), q(x) \in P_2$ we have

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx = \int_{-1}^{1} q(x)p(x)dx = \langle q(x), p(x) \rangle.$$

(ii) For all $p(x), q(x), r(x) \in P_2$ we have

$$\begin{aligned}\langle p(x)+q(x), r(x)\rangle &= \int_{-1}^{1}(p(x)+q(x))r(x)dx \\ &= \int_{-1}^{1}p(x)r(x)dx + \int_{-1}^{1}q(x)r(x)dx \\ &= \langle p(x), r(x)\rangle + \langle q(x), r(x)\rangle.\end{aligned}$$

(iii) For all $p(x), q(x) \in P_2$, $\lambda \in \mathbb{R}$ we have

$$\langle \lambda p(x), q(x)\rangle = \int_{-1}^{1}\lambda p(x)q(x)dx = \lambda\int_{-1}^{1}p(x)q(x)dx = \lambda\langle p(x), q(x)\rangle.$$

(iv) For all $p(x) \in P_2$ we have $\langle p(x), p(x)\rangle = \int_{-1}^{1}(p(x))^2dx \geq 0$ as it is the integral of a non-negative polynomial. Moreover, this integral can only be zero if $p(x) = 0$ the zero polynomial.

[6]

(b) The norm of a polynomial $p(x)$ is given by $||p(x)|| = \sqrt{\int_{-1}^{1}(p(x))^2dx}$. $||x||^2 = \int_{-1}^{1}x^2dx = \frac{2}{3}$ so $||x|| = \frac{\sqrt{2}}{\sqrt{3}}$. [2]

(c) Two polynomials $p(x)$ and $q(x)$ are orthogonal if and only if $\langle p(x), q(x)\rangle = \int_{-1}^{1}p(x)q(x)dx = 0$. The two polynomials $x+1$ and x^2 are not orthogonal as

$$\langle (x+1), x^2\rangle = \int_{-1}^{1}(x+1)x^2dx = \frac{2}{3} \neq 0.$$

[2]

(d) A set of polynomials is orthonormal if they are pairwise orthogonal and all have norm 1. This set is orthonormal as

$$\langle p_1(x), p_2(x)\rangle = \int_{-1}^{1}\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{2}}xdx = 0,$$

$$||p_1(x)||^2 = \int_{-1}^{1}\frac{1}{2}dx = 1 \quad \text{and} \quad ||p_2(x)||^2 = \int_{-1}^{1}\frac{3}{2}x^2dx = 1.$$

[4]

(e) First define

$$q(x) = x^2 - \langle x^2, p_1(x)\rangle p_1(x) - \langle x^2, p_2(x)\rangle p_2(x).$$

As

$$\langle x^2, p_1(x)\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 dx = \frac{2}{3\sqrt{2}},$$
$$\langle x^2, p_2(x)\rangle = \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} x^3 dx = 0,$$

we have $q(x) = x^2 - \frac{2}{3\sqrt{2}}\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$. Now

$$||q(x)||^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx = \frac{8}{45},$$

so $||q(x)|| = \frac{2\sqrt{2}}{3\sqrt{5}}$. Now we take

$$p_3(x) = \frac{3\sqrt{5}}{2\sqrt{2}} q(x) = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}).$$

[6]