3 Differential equations

In this last part of the Calculus course we are going to study some new methods to solve certain types of differential equations. This will be a continuation of what you studied in your 1st year calculus course and, as in there, we are going to deal exclusively with **real functions** $y = f(x)$ **of one real variable**. As last year, we will concentrate on a subclass of differential equations, that is **linear differential equations**.

3.1 Linear differential equations

Definition: A **n-th order linear differential equation** is an equation of the form

$$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = R(x), \tag{3.1}$$

where

$$y^{(k)} = \frac{d^k y}{dx^k}, \tag{3.2}$$

is the k-th derivative of the function y with respect to the variable x and $P_1(x), \dots P_n(x)$ and $R(x)$ are real functions of x which are continuous in a certain interval $I \in \mathbb{R}$. The **order** of the differential equation is the highest order of the derivatives appearing in it (in this case n). The equation is **linear** because no terms involving products of y and its derivatives appear (e.g. terms like $yy^{(k)}$ or $y^{(p)}y^{(k)}$).

Notation: In order to prove certain general properties of linear differential equations it is convenient to introduce various notations. For example, we can introduce an operator L which acts on a function $f(x)$ as follows

$$L(f) = f^{(n)} + P_1(x)f^{(n-1)} + P_2(x)f^{(n-2)} + \cdots + P_n(x)f, \tag{3.3}$$

that is,

$$L = D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \cdots + P_n(x), \tag{3.4}$$

with

$$D = \frac{d}{dx} \qquad \text{and} \qquad D^n = \frac{d^n}{dx^n}. \tag{3.5}$$

In terms of the operator L, the equation (3.1) would take the form

$$L(y) = R(x) = (D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \cdots + P_n(x))y. \tag{3.6}$$

Using the definition of the operator L we can easily deduce the two following properties:

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad \text{and} \qquad L(\alpha y) = \alpha L(y), \tag{3.7}$$

which follow from the properties of the derivative. These two properties are equivalent to saying that L is a **linear operator**.

3.1.1 Second order linear differential equations

Let us now consider a particular case of the equations (3.1), that is 2nd order differential equations

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y). \tag{3.8}$$

As usual in this context, solving these equations is done in two steps: fist we must solve the homogeneous 2nd-order differential equation

$$y'' + P_1(x)y' + P_2(x)y = 0 = L(y), \tag{3.9}$$

and then find a particular solution of the inhomogeneous equation (3.8). In the next sections we are going to see why this is the case and how to solve (3.8)-(3.9).

Homogeneous equations: In this section we are going to study how to solve homogeneous equations such as (3.9). We will also see two theorems which will answer two fundamental questions: when do solutions to (3.9) exist? and under which conditions is the solution of (3.9) unique? Before entering these details, let us look at a very simple example:

Example: Consider the following homogeneous 2nd order differential equation

$$y'' + k^2 y = 0, \tag{3.10}$$

where $k \neq 0$ is a constant. This is an equations which you have learnt how to solve last year. In general you try solutions of the type:

$$y = Ce^{mx}, \tag{3.11}$$

with C and m being constants. Then you plug this solution into the equation above and obtain

$$m^2 + k^2 = 0 \qquad \Rightarrow \qquad m = \pm ik. \tag{3.12}$$

Therefore the most general solution of (3.10) is of the form

$$y = C_1 e^{ikx} + C_2 e^{-ikx}, \tag{3.13}$$

with C_1, C_2 being generic constants. Equivalently we can write

$$y = A_1 \sin(kx) + A_2 \cos(kx), \tag{3.14}$$

with $C_1 = (A_2 - iA_1)/2$ and $C_2 = (A_2 + iA_1)/2$.

The solution (3.1.1) with A_1, A_2 arbitrary is the most general solution of (3.10). However, in many problems we have to select a particular solution from the set (3.1.1), a solution satisfying certain additional conditions. There conditions are called **initial conditions**. For example, suppose we want to find a solution of (3.10) such that for a certain value of $x = x_0$,

$$y'(x_0) = b \qquad \text{and} \qquad y(x_0) = a. \tag{3.15}$$

This means that the following two equations have to be satisfied

$$y(x_0) = a = A_1 \sin(kx_0) + A_2 \cos(kx_0), \tag{3.16}$$
$$y'(x_0) = b = A_1 k \cos(kx_0) - A_2 k \sin(kx_0). \tag{3.17}$$

This is a system of two linear equations for two unknowns. An equivalent way of writing these equations is

$$\begin{pmatrix} \sin(kx_0) & \cos(kx_0) \\ k\cos(kx_0) & -k\sin(kx_0) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.18}$$

A system of equations like this can **always** be solved (we have 2 equations and 2 unknowns). It will have a **unique solution** provided the two equations (3.17) are independent from each other. Another way of saying that is to say that the determinant

$$\begin{vmatrix} \sin(kx_0) & \cos(kx_0) \\ k\cos(kx_0) & -k\sin(kx_0) \end{vmatrix} = k(\cos^2(kx_0) + \sin^2(kx_0)) = k, \tag{3.19}$$

must be non-vanishing ($k \neq 0$). Since it was assumed from the beginning that $k \neq 0$, we can conclude that there is a unique solution of the initial value problem

$$y'' + k^2 y = 0 \quad \text{with} \quad y'(x_0) = b \quad \text{and} \quad y(x_0) = a. \tag{3.20}$$

Existence theorem: Let $P_1(x), P_2(x)$ be continuous functions of x in an open interval I and let

$$L(y) = y'' + P_1(x)y' + P_2(x)y. \tag{3.21}$$

If $x_0 \in I$ and a, b are given real numbers, there exists $y = f(x)$ such that $L(y) = 0$ on I with $f(x_0) = a$ and $f'(x_0) = b$.

Uniqueness theorem: Let

$$L(y) = y'' + P_1(x)y' + P_2(x)y, \tag{3.22}$$

and let $f(x), g(x)$ be solutions of the homogenous equation $L(f) = L(g) = 0$ on an open interval I of $\mathbb{R}$. Assume that $f(x_0) = g(x_0)$ and $f'(x_0) = g'(x_0)$ for some $x_0 \in I$. Then $f(x) = g(x)$ for all $x \in I$.

Theorem (characterization of solutions): Let

$$L(y) = 0 = y'' + P_1(x)y' + P_2(x)y, \tag{3.23}$$

be a 2nd order linear and homogeneous differential equation with coefficients $P_1(x)$, $P_2(x)$ which are continuous in an open interval $I \in \mathbb{R}$. Let $u_1(x), u_2(x)$ be two non-zero functions satisfying $L(u_1) = L(u_2) = 0$ in I and such that $u_2(x)/u_1(x)$ **is not a constant in** I. Then

$$y = c_1 u_1(x) + c_2 u_2(x), \tag{3.24}$$

is a solution of $L(y) = 0$ on I. Conversely, if y is a solution of (3.23) in I, then there exist constants c_1, c_2 such that (3.24) holds. This means that all solutions of (3.23) are of the form (3.24).

Proof:

(a) To proof that $c_1u_1(x)+c_2u_2(x)$ is a solution of (3.23) provided that $u_1(x), u_2(x)$ are solutions of (3.23) we only have to use the first property in (3.7), that is the fact that L is a linear operator. Therefore

$$L(c_1u_1(x) + c_2u_2(x)) = c_1L(u_1(x)) + c_2L(u_2(x)). \tag{3.25}$$

(b) The second statement we have to proof is that all solutions of (3.23) are of the form $y = c_1u_1(x) + c_2u_2(x)$. To prove that we can consider a solution of the equation $y = f(x)$ and choose a point $x_0 \in I$ with initial conditions $y'(x_0) = f'(x_0)$ and $y(x_0) = f(x_0)$. If we are able to show that we can find two constants c_1, c_2 such that

$$\begin{aligned} f(x_0) &= c_1u_1(x_0) + c_2u_2(x_0), & (3.26)\\ f'(x_0) &= c_1u_1'(x_0) + c_2u_2'(x_0), & (3.27)\end{aligned}$$

then we would have proven that f and $c_1u_1(x) + c_2u_2(x)$ have the same values and the same derivatives at the point x_0. Then we can use the uniqueness theorem to conclude that they are the same function

$$f(x) = c_1u_1(x) + c_2u_2(x). \tag{3.28}$$

To show the existence of c_1, c_2 such that (3.27) holds we only need to prove that there is at least a point $x_0 \in I$ such that the system of equations (3.27) has a solution. As in the example, this is equivalent to finding at least a point $x_0 \in I$ such that the determinant

$$\begin{vmatrix} u_1(x_0) & u_2(x_0) \\ u_1'(x_0) & u_2'(x_0) \end{vmatrix} = u_1(x_0)u_2'(x_0) - u_1'(x_0)u_2(x_0) \neq 0. \tag{3.29}$$

In general, we define the determinant

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = u_1(x)u_2'(x) - u_1'(x)u_2(x), \tag{3.30}$$

and we call it the **Wronskian**.

Proving that there is at least a point $x_0 \in I$ such that the Wronskian $W(x_0) \neq 0$ is equivalent to proving that all solutions of the homogeneous equation (3.23) have the form (3.24). We will carry out this proof by proving that if we assume the contrary we arrive to a contradiction: Suppose that $W(x) = 0$ for all $x \in I$. Then this will imply that

$$\left(\frac{u_2}{u_1}\right)' = \frac{u_1u_2' - u_2u_1'}{u_1^2} = \frac{W}{u_1^2} = 0, \tag{3.31}$$

but this is in contradiction with one of the assumptions we made at the beginning, namely that u_2/u_1 is not a constant for $x \in I$. This means that the derivative of u_2/u_1 can not be vanishing, which is what would happen if the Wronskian is zero everywhere. Therefore $W(x_0) \neq 0$ for at least one point $x_0 \in I$.

Corollary: This theorem tells us that all solutions of the equation $L(y) = 0$ are of the form $c_1u_1(x) + c_2u_2(x)$, with c_1, c_2 being arbitrary constants. For this reason

$$y = c_1u_1(x) + c_2u_2(x), \tag{3.32}$$

is called the general solution of $L(y) = 0$. It follows also from the theorem that we can find the general solution by finding two particular solutions $u_1(x), u_2(x)$ such that $u_2/u_1 \neq$ constant. If $u_2/u_1 \neq$ constant we call u_1, u_2 **linearly independent solutions.**

Inhomogeneous equations:
Consider now the inhomogeneous equation

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y), \tag{3.33}$$

with P_1, P_2, R being continuous functions on an open interval $I \in \mathbb{R}$. Suppose that y_1 and y_2 are two solutions of the inhomogeneous equation

$$L(y_1) = L(y_2) = R(x), \tag{3.34}$$

then by linearity of L

$$L(y_1 - y_2) = L(y_1) - L(y_2) = 0, \tag{3.35}$$

therefore $y_1 - y_2$ is a solution of the homogeneous equation $L(y) = 0$. However the previous theorem told us that the solutions of the homogeneous equation always are of the form $c_1u_1(x) + c_2u_2(x)$. Therefore $y_1 - y_2$ must be of the form

$$y_1 - y_2 = c_1u_1(x) + c_2u_2(x), \tag{3.36}$$

for some c_1, c_2, or equivalently

$$y_1 = c_1u_1(x) + c_2u_2(x) + y_2. \tag{3.37}$$

This proves that any pair of solutions of the inhomogeneous equation are related by (3.37). In other words, given a particular solution of $L(y) = R$, say y_1, all solutions of the differential equation are contained in the set

$$y = y_1 + c_1u_1(x) + c_2u_2(x), \tag{3.38}$$

where $L(u_1) = L(u_2) = 0$ and c_1, c_2 are arbitrary constants. For that reason (3.38) is called the **general solution** of the inhomogeneous equation (3.33).

Conclusion: Given an inhomogeneous equation

$$y'' + P_1(x)y' + P_2(x)y = R(x) = L(y), \tag{3.39}$$

with P_1, P_2, R being continuous functions on an open interval $I \in \mathbb{R}$, there are two problems we need to solve in order to find its general solution:

1. Find the general solution of the homogeneous equation $L(y) = 0$, i.e. $c_1u_1(x) + c_2u_2(x)$.

2. Find a particular solution of the inhomogeneous equation $L(y) = R$.

From last year's Calculus you already know some methods to solve simple homogeneous equations. Therefore we are going to start by learning a method to solve inhomogeneous equations: the method of variation of parameters.

3.1.2 The method of variation of parameters

Theorem: Let u_1, u_2 be two linearly independent solutions of a homogeneous equation $L(y) = 0$ on an interval I, with

$$L(y) = y'' + P_1(x)y' + P_2(x)y. \tag{3.40}$$

Let

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x), \tag{3.41}$$

be the Wronskian of u_1 and u_2. Then, the inhomogeneous equation $L(y) = R(x)$ has a solution y_1 given by

$$y_1(x) = v_1(x)u_1(x) + v_2(x)u_2(x), \tag{3.42}$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx, \tag{3.43}$$

Proof: Let us suppose that $y = v_1(x)u_1(x) + v_2(x)u_2(x)$ and try to determine $v_1(x)$ and $v_2(x)$. Then we just have to plug this solution into the equation

$$R(x) = y'' + P_1(x)y' + P_2(x)y, \tag{3.44}$$

and see what conditions must v_1, v_2 satisfy. Let us first compute

$$\begin{aligned} y' &= v_1'(x)u_1(x) + v_1(x)u_1'(x) + v_2'(x)u_2(x) + v_2(x)u_2'(x), & (3.45)\\ y'' &= v_1''(x)u_1(x) + v_1'(x)u_1'(x) + v_1'(x)u_1'(x) + v_1(x)u_1''(x) \\ &+ v_2''(x)u_2(x) + v_2'(x)u_2'(x) + v_2'(x)u_2'(x) + v_2(x)u_2''(x) \\ &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + v_2(x)u_2''(x), & (3.46) \end{aligned}$$

substituting these derivatives into (3.44) we obtain

$$\begin{aligned} R(x) &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_1(x)u_1''(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + v_2(x)u_2''(x) + P_1(x)\left(v_1'(x)u_1(x) + v_1(x)u_1'(x)\right. \\ &+ \left. v_2'(x)u_2(x) + v_2(x)u_2'(x)\right) + P_2(x)(v_1(x)u_1(x) + v_2(x)u_2(x)), \end{aligned} \tag{3.47}$$

now we can use the fact that u_1, u_2 are solutions of the homogeneous equation, that is, they satisfy

$$\begin{aligned} 0 &= u_1''(x) + P_1(x)u_1'(x) + P_2(x)u_1(x), & (3.48)\\ 0 &= u_2''(x) + P_1(x)u_2'(x) + P_2(x)u_2(x). & (3.49) \end{aligned}$$

Using these equations we can prove that all terms in (3.47) which are proportional to $v_1(x)$ and $v_2(x)$ vanish, leaving us with the following condition

$$\begin{aligned} R(x) &= v_1''(x)u_1(x) + 2v_1'(x)u_1'(x) + v_2''(x)u_2(x) \\ &+ 2v_2'(x)u_2'(x) + P_1(x)\left(v_1'(x)u_1(x) + v_2'(x)u_2(x)\right) \\ &= (v_1'(x)u_1'(x) + v_2'(x)u_2'(x)) + (v_1'(x)u_1(x) + v_2'(x)u_2(x))' \\ &+ P_1(x)\left(v_1'(x)u_1(x) + v_2'(x)u_2(x)\right). \end{aligned} \tag{3.50}$$

One way of solving this equation is to impose further conditions on v_1, v_2 (remember, that we only need to obtain a particular solution of the inhomogeneous equation). For example, let us look for v_1, v_2 satisfying

$$v_1'(x)u_1(x) + v_2'(x)u_2(x) = 0 \quad \text{and} \quad v_1'(x)u_1'(x) + v_2'(x)u_2'(x) = R(x). \tag{3.51}$$

In this case we can solve these equations for v_1', v_2'. From the first equation we obtain

$$v_1'(x) = -v_2'(x)\frac{u_2(x)}{u_1(x)}, \tag{3.52}$$

substituting this in the second equation we get

$$\begin{aligned} &-u_1'(x)v_2'(x)\frac{u_2(x)}{u_1(x)} + v_2'(x)u_2'(x) = R(x) \\ \Rightarrow \quad &v_2'(x) = \frac{u_1(x)R(x)}{u_2'(x)u_1(x) - u_1'(x)u_2(x)} = \frac{u_1(x)R(x)}{W(x)}. \end{aligned} \tag{3.53}$$

Substituting this back into (3.52) we obtain

$$v_1'(x) = -\frac{u_2(x)R(x)}{W(x)}. \tag{3.54}$$

Integrating these equations we obtain the solutions (3.43). Let us see how this works with some examples:

Example 1: Solve the following 2nd order linear differential equation:

$$y'' + y = \tan x. \tag{3.55}$$

First we have to solve the homogeneous equation:

$$y'' + y = 0. \tag{3.56}$$

Last year you have seen that this kind of homogeneous equations with constant coefficients can always be solved by looking for solutions of the type $y = ce^{mx}$. Substituting this solution in (3.56) we obtain

$$m^2 + 1 = 0 \qquad \Rightarrow \qquad m = \pm i, \tag{3.57}$$

which means that the general solution of (3.56) can be written as

$$y = c_1 \cos x + c_2 \sin x, \tag{3.58}$$

therefore we identify

$$u_1(x) = \cos x \qquad \text{and} \qquad u_2(x) = \sin x. \tag{3.59}$$

Having this solution the next step is to solve the inhomogeneous equation (3.55). To do that we can use the method of variation of parameters which tells us that a particular solution of the inhomogeneous equation is given by:

$$y = v_1(x)u_1(x) + v_2(x)u_2(x) = v_1(x)\cos x + v_2(x)\sin x, \tag{3.60}$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx \qquad \text{and} \qquad v_2(x) = \int u_1(x)\frac{R(x)}{W(x)}dx. \tag{3.61}$$

In our case

$$\begin{aligned} R(x) &= \tan x, \qquad (3.62)\\ W(x) &= \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1. \end{aligned}$$

Therefore we have

$$v_1(x) = -\int \sin x \tan x dx = -\int \frac{\sin^2 x}{\cos x}dx, \tag{3.63}$$

we can solve this integral by changing variables as

$$t = \sin x \qquad \Rightarrow \qquad dt = \cos x dx, \tag{3.64}$$

which allows us to write

$$\begin{aligned} v_1(x) &= -\int \frac{t^2}{1-t^2}dt = -\int \frac{t^2 - 1 + 1}{1-t^2}dt \\ &= \int dt - \int \frac{1}{1-t^2}dt = t + \frac{1}{2}\int \left[\frac{1}{1-t} - \frac{1}{1+t}\right] dt \\ &= t + \frac{1}{2}\ln\left[\frac{t-1}{t+1}\right] + C, \end{aligned} \tag{3.65}$$

Recalling that $t = \sin x$ we obtain

$$v_1(x) = \sin x + \frac{1}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right] + C. \tag{3.66}$$

Similarly

$$v_2(x) = \int \cos x \tan x dx = \int \sin x dx = -\cos x + C' \tag{3.67}$$

Therefore, the general solution of (3.55) is

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + \left[\sin x + \frac{1}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right]\right]\cos x - \cos x \sin x \\ &= c_1 \cos x + c_2 \sin x + \frac{\cos x}{2}\ln\left[\frac{\sin x - 1}{\sin x + 1}\right]. \end{aligned} \tag{3.68}$$

Notice that in the final solution we can forget about the constants C, C' because we are just interested in a particular solution of the inhomogeneous equation. Therefore, we can just choose them to be zero!

Example 2: Solve the equation

$$y'' + y = x^3. \tag{3.69}$$

You can solve this sort of equation without using the method of variation of parameters. In fact you have already solved equations of this type last year. Remember that whenever you have an equation of the form

$$y'' + ay' + by = R(x), \tag{3.70}$$

where a, b are constant coefficients and $R(x)$ is a polynomial of degree n, the solution to this equation is always of the form

$$y = a_1 + a_2 x + \ldots + a_n x^n, \tag{3.71}$$

with $a_1, a_2, \ldots, a_n$ constants.
Therefore a particular solution of (3.69) will be of the type

$$y = Ax^3 + Bx^2 + Cx + D, \tag{3.72}$$

with derivatives

$$\begin{aligned} y' &= 3Ax^2 + 2Bx + C, && (3.73)\\ y'' &= 6Ax + 2B, && (3.74) \end{aligned}$$

and if we substitute (3.72) and (3.74) in (3.69) we obtain

$$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3, \tag{3.75}$$

from where it follows,

$$A = 1, B = 0, \quad 6A + C = 0 \Rightarrow C = -6, \quad B + 2D = 0 \Rightarrow D = 0. \tag{3.76}$$

So, our particular solution of the inhomogeneous equation is

$$y = x^3 - 6x, \tag{3.77}$$

and the general solution of the homogeneous equation is the same as in example 1. Therefore the general solution of (3.69) is

$$y = x^3 - 6x + c_1 \cos x + c_2 \sin x. \tag{3.78}$$

We could have solved this problem by using the method of variation of parameters. In that case the particular solution of the inhomogeneous equation would be

$$y = v_1(x) \cos x + v_2(x) \sin x, \tag{3.79}$$

with

$$\begin{aligned} v_1(x) &= -\int u_2(x) \frac{R(x)}{W(x)} dx = -\int x^3 \sin x, \\ v_2(x) &= \int u_1(x) \frac{R(x)}{W(x)} dx = \int x^3 \cos x. && (3.80) \end{aligned}$$

These integrals can be solved by parts (in fact we have to use integration by parts three times to solve each of the integrals!)

$$\begin{aligned} \int x^3 \sin x &= -x^3 \cos x + 3 \int x^2 \cos x dx \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \int \cos x dx \\ &= (6x - x^3) \cos x + (3x^2 - 6) \sin x, && (3.81) \end{aligned}$$

$$\begin{aligned}
\int x^3 \cos x &= x^3 \sin x - 3\int x^2 \sin x dx \\
&= x^3 \sin x + 3x^2 \cos x - 6\int x \cos x dx \\
&= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6\int \sin x dx \\
&= (x^3 - 6x)\sin x + (3x^2 - 6)\cos x. \qquad (3.82)
\end{aligned}$$

Therefore the particular solution of the inhomogeneous equation would be

$$\begin{aligned}
y &= v_1(x)\cos x + v_2(x)\sin x = -((6x - x^3)\cos x + (3x^2 - 6)\sin x)\cos x \\
&\quad +((x^3 - 6x)\sin x + (3x^2 - 6)\cos x)\sin x \\
&= (x^3 - 6x)(\cos^2 x + \sin^2 x) = x^3 - 6x. \qquad (3.83)
\end{aligned}$$

Therefore, we obtain the same solution as with the other method but now we have to compute two quite lengthy integrals! The conclusion from this problem is that we must only use the method of variation of parameters when no other simpler method works. The method of variation of parameters is very powerful since it works for cases in which all other methods we know fail but we should not use it if we do not need to.