X3 Fluid Dynamics **Question Sheet 1**

Answer all questions. Solutions to be handed in to SEMS undergraduate office by 4:00pm on Monday 15th November 2010

The Navier-Stokes equation for an incompressible ($\nabla \cdot \mathbf{u} = 0$) Newtonian fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}.$$

Euler's equation for incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla \Phi$ and $\mathbf{u} = \nabla \phi$.

1. Find the equations of the streamlines of the flow when

   (i) $\mathbf{u} = (Ax, -Ay, 0)$

   (ii) $\mathbf{u} = (Ax/2, Ay/2, -Az)$

   (iii) $\mathbf{u} = (2x - 3y, 4x/3 - 2y, 0)$

   (iv) $\mathbf{u} = \left(\frac{A}{a} \cos ax \sin by, -\frac{A}{b} \sin ax \cos by, 0 \right)$

   In each case attempt to sketch the streamlines, and check to see if the flow is incompressible. [Solutions to (i) and (ii) at end.]

2. Repeat the derivation of the phase velocity for small waves on water, but assume that the water is now of finite depth h. Hence if the mean surface level is at $z = 0$ then the potential for the velocity must satisfy

   $$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h.$$

   Show that as the waves become longer the phase velocity and group velocity converge to some finite value which is to be determined.

If surface tension is important then the method has to be adapted slightly. Instead of the pressure in the water being p_0 at the surface it will be given by

$$p = p_0 - T\frac{\partial^2 \eta}{\partial x^2}$$

where $\eta(x,t)$ is the surface elevation. Find the modified dispersion relation (the expression for ω as a function of k) when surface tension is important.

3. Find the potentials and streamfunctions of the flows corresponding to the following complex potential, and make sketches of the flows.

$$w(z) = (1+i)z, \qquad \text{(a)} \qquad\qquad w(z) = (1+i)z^2, \qquad \text{(b)}$$

$$w(z) = z^2 - z, \qquad \text{(c)} \qquad\qquad w(z) = e^{iz}, \qquad \text{(d)}$$

$$w(z) = \cos(\pi z), \qquad \text{(e)} \qquad\qquad w(z) = \ln\left(\frac{z+1}{z-1}\right). \qquad \text{(f)}$$

4. An inviscid fluid undergoing a 2-dimensional flow occupies the region $x \geq 0$. At $x = 0$ there is a plane rigid wall. A line vortex of strength Γ is located in the flow at $z = x_0 + iy_0$ with x_0 and y_0 real. Explain why the complex potential for the flow is

$$w(z) = -\frac{i\Gamma}{2\pi}\ln(z - x_0 - iy_0) + \frac{i\Gamma}{2\pi}\ln(z + x_0 - iy_0).$$

Show that the vortex moves parallel to the wall in the negative y-direction with speed $\Gamma/4\pi x_0$.

If $y_0 = 0$, show that the velocity of the fluid at the wall is given by

$$v = -\frac{\Gamma x_0}{\pi(y^2 + x_0^2)},$$

and show that at the wall $x = 0$,

$$\frac{\partial \phi}{\partial t} = -\frac{\Gamma^2}{4\pi^2(y^2 + x_0^2)}.$$

Hence, using Bernoulli's equation, show that the net force exerted on the wall by the vortex is

$$\int_{-\infty}^{\infty} p\,\mathrm{d}y = 0.$$

You may quote the results that

$$\int_{-\infty}^{\infty} \frac{1}{y^2 + a^2}\,dy = \frac{\pi}{a} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{(y^2 + a^2)^2}\,dy = \frac{\pi}{2a^3}.$$

5. Take photos of (a) the pattern of ripples caused by a stone falling in water and (b) the wake of a boat, duck or similar object moving through water. Email these to `o.s.kerr@city.ac.uk`. Make notes of what features you see that match with theory. Bonus points may be given for any other photos of interesting behaviour of fluids.

Solution (you still have to do them yourselves, sketch them and hand them in!)

1. For these use use the general formula

$$\frac{\frac{dx}{ds}}{u} = \frac{\frac{dy}{ds}}{v} = \frac{\frac{dz}{ds}}{w},$$

where s is some measure along the streamline. For 2-dimensional problems this can be re-expressed as

$$\frac{dy}{dx} = \frac{v}{u}.$$

(i) $\dfrac{dy}{dx} = -\dfrac{y}{x} \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \;\Rightarrow\; \ln y = -\ln x + C$ or $xy = C'$

(ii) $\mathbf{u} = (Ax/2, Ay/2, -Az)$ If you look at flows in planes of constant z you get $\dfrac{dy}{dx} = \dfrac{y}{x} \;\Rightarrow\; y = cx$, i.e., straight lines through the origin. In the plane, for example, $y = 0$, you get $\dfrac{dz}{dx} = -\dfrac{2z}{x} \;\Rightarrow\; \displaystyle\int \frac{dz}{z} = -2\int \frac{dx}{x} \;\Rightarrow\; \ln|z| = -2\ln|x| + C \;\Rightarrow\; zx^2 = C'$. Three dimensional flows are always difficult to plot....