Fluid Dynamics

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Small waves on deep water travelling with speed c and with wavenumber k have a velocity given by

$$\mathbf{u}(\mathbf{x}, t) = k\epsilon \left(\cos k(x - ct), \sin k(x - ct)\right) e^{ky}$$

We will derive this later in the course.

wavenumber: $k = 2\pi/L$ where L is the wavelength.

wave speed: we will see later $c = (g/k)^{1/2}$ where g is the acceleration due to gravity.

wave amplitude: The wave amplitude, ϵ, is small.

Streamlines

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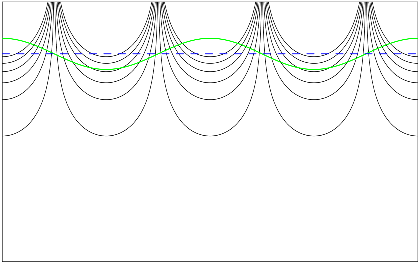
so

$$y = -\frac{1}{k} \ln(\cos k(x - ct)) + C$$



Average velocity

The average velocity at a given point is

$$\frac{1}{T}\int_0^T \mathbf{u}\,dt \to 0 \qquad \text{as} \qquad T \to \infty$$

What is the average velocity of a fluid particle?

$$\frac{dx}{dt} = u(x, y, t) = k\epsilon \cos k(x - ct)\, e^{ky}$$
$$\frac{dy}{dt} = v(x, y, t) = k\epsilon \sin k(x - ct)\, e^{ky}$$

If ϵ is small then particles do not move much and are approximately constant. So "guess"

$$x(t) = x_0 + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$$

$$y(t) = y_0 + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots$$

the leading order terms are all of size ϵ:

$$\frac{dx_1}{dt} = k \cos k(x_0 - ct)\, e^{ky_0}$$

$$\frac{dy_1}{dt} = k \sin k(x_0 - ct)\, e^{ky_0}$$

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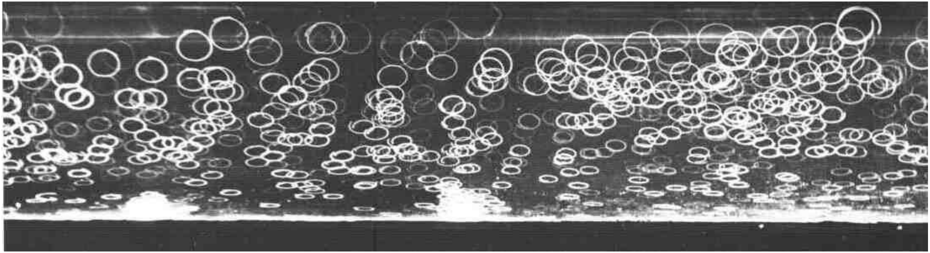
These have the solutions

$$x_1(t) = -\frac{1}{c}\sin k(x_0 - ct)\, e^{ky_0}$$

$$y_1(t) = \frac{1}{c}\cos k(x_0 - ct)\, e^{ky_0}$$

At this order we have particles going around in circles of radius $\epsilon e^{ky_0}/c$.

Waves travelling in from the right



Wallet & Ruellan, 1950

Note: Near the bottom of tank the trajectories become more elliptical.

Drift

But the solution is not exact. Putting our expansion into the governing equation, we get

$$\epsilon \frac{dx_1}{dt} + \epsilon^2 \frac{dx_2}{dt} + \cdots = \epsilon k \cos k((x_0 + \epsilon x_1 + \cdots) - ct)\, e^{k(y_0 + \epsilon y_1 + \cdots)}$$

If we want to get the solution accurate to order ϵ^2 then we take Taylor series of the right hand side. If we expand the right-hand side in powers of ϵ then we should be able to equate the terms proportional to ϵ, the terms proportional to ϵ^2, and so on.

To find the higher order terms we observe

$$e^{k(y_0+\epsilon y_1+\cdots)} = e^{ky_0} e^{\epsilon k y_1} e^{\epsilon^2 k y_2} \dots$$

$$= e^{ky_0}(1+\epsilon k y_1 + \epsilon^2 k^2 y_1^2/2 + \cdots)(1+\epsilon^2 k y_2 + \cdots)\dots$$
$$= e^{ky_0} + \epsilon k y_1 e^{ky_0} + \epsilon^2(k^2 y_1^2/2 + k y_2) e^{ky_0} + \cdots$$

and

$$\cos k(x_0+\epsilon x_1+\cdots - ct) = \cos k(x_0 - ct) - \epsilon k x_1 \sin k(x_0-ct) + \cdots$$

If we equate terms of order ϵ we get the equations we had before.

If we equate terms of order ϵ^2 we get

$$\frac{dx_2}{dt} = e^{ky_0}\left(k^2 y_1 \cos k(x_0 - ct) - k^2 x_1 \sin k(x_0 - ct)\right)$$

Substituting in the solutions for x_1 and y_1 gives

$$\frac{dx_2}{dt} = e^{ky_0}\left(\frac{k^2}{c} e^{ky_0} \cos^2 k(x_0 - ct) + \frac{k^2}{c} e^{ky_0} \sin^2 k(x_0 - ct)\right) = \frac{k^2}{c} e^{2ky_0}$$

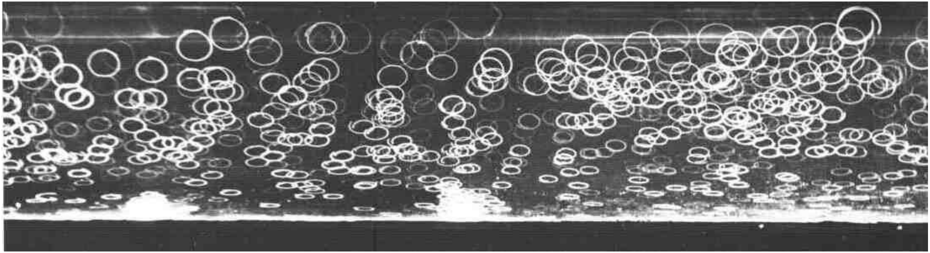
and so

$$x_2 = \frac{k^2 t}{c} e^{2ky_0}$$

Since

$$x_2 = \frac{k^2 t}{c} e^{2ky_0}$$

particles drift to the right with speeds that depend on the depth.



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