The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Gravity waves on the surface of a fluid can be modified by the effects of surface tension.

   The surface of an inviscid irrotational fluid is given by

   $$y = \eta(x,t) = \epsilon\cos(kx - \omega t).$$

   Find the potential of the flow assuming the fluid is infinitely deep in the negative y-direction and that ϵ is small and so terms proportional to ϵ^2 and smaller can be ignored.

   The pressure of the fluid at the surface is given by

   $$p = P_0 - T\frac{\partial^2\eta}{\partial^2 x}$$

   where T is a positive constant. Including the effect of gravity, so the gravitational potential is $\Phi = gz$, find a relation between the frequency ω and the wavenumber k.

   Determine the group velocity of the waves, and show that it has a minimum for some wavenumber, k, which is to be determined.

Turn over ...

2. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

An incompressible flow is given by

$$\mathbf{u}(x, y, z, t) = \left(-Ax + u'(x, y, t),\ -Ay + v'(x, y, t),\ 2Az\right),$$

where A is constant and $(u', v', 0)$ is a perturbation velocity. Show that the vorticity of this flow is of the form $\boldsymbol{\omega} = (0,\ 0,\ \omega_3(x, y, t))$ where ω_3 satisfies

$$\frac{\partial \omega_3}{\partial t} - Ax\frac{\partial \omega_3}{\partial x} - Ay\frac{\partial \omega_3}{\partial y} + u'\frac{\partial \omega_3}{\partial x} + v'\frac{\partial \omega_3}{\partial y} = 2A\omega_3 + \nu\left(\frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2}\right).$$

If cylindrical polar coordinates are used, with $x = r\cos\theta$ and $y = r\sin\theta$ and the radial azimuthal and axial components of the perturbation velocity are given by u_r, u_θ and u_z respectively, then this can be written as

$$\frac{\partial \omega_3}{\partial t} - Ar\frac{\partial \omega_3}{\partial r} + u_r\frac{\partial \omega_3}{\partial r} + \frac{u_\theta}{r}\frac{\partial \omega_3}{\partial \theta} = 2A\omega_3 + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_3}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_3}{\partial \theta^2}\right).$$

Find the partial differential equation that $f(r, t)$ must satisfy if $u_r = 0$ and the vorticity is given by

$$\omega_3 = f(r, t).$$

(i) Show that if $f(r, t)$ is independent of r, i.e., $f(r, t) = g(t)$ for some function $g(t)$, then the vorticity grows exponentially.

(ii) Show that if $f(r, t)$ is independent of t, i.e., $f(r, t) = h(r)$ for some function $h(r)$, then $h(r)$ satisfies

$$Ar^2 h + \nu r\frac{\mathrm{d}h}{\mathrm{d}r} = B,$$

where B is a constant. Why must B be zero?

Find the general solution to this differential equation with $B = 0$.

[You may quote the result $\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u}$.]

Turn over ...

3. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder.

Show how the Milne-Thompson theorem can be used to derive the complex potential for the uniform flow past a circle of radius a centred at the origin with the flow in the far field inclined at an angle α to the x-axis.

If a circulation Γ around the cylinder is added the complex potential becomes

$$w(z) = U\left(e^{-i\alpha}z + \frac{a^2 e^{+i\alpha}}{z}\right) - \frac{i\Gamma}{2\pi}\ln z,$$

where the far-field flow is inclined at an angle α to the x-axis. Verify that the mapping

$$\zeta = z + \frac{a^2}{z}$$

maps the circle $z = ae^{i\theta}$ in the z-plane onto the straight line between $\zeta = -2a$ and $\zeta = 2a$ in the ζ-plane. Hence write down the potential for the flow past a flat plate with circulation in the ζ-plane.

Find the value of the circulation, Γ, such that the singularity in the velocity at the trailing edge $\zeta = 2a$ is removed.

4. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

A plate lies along the positive x-axis. There is an exterior flow past the plate given by

$$U(x) = Cx^{1/2}.$$

By looking for a solution to the boundary layer equations in terms of the stream function ψ which takes the form

$$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

show that the problem can be reduced to the ordinary differential equation

$$f''' + ff'' + \frac{2}{3}\left(1 - f'^2\right) = 0,$$

where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

5. An incompressible viscous fluid flows down a flat slope under the force of gravity. What are the boundary conditions for the fluid at the point of contact with the slope and at the free surface?

   Using orthogonal coordinates with the x-axis pointing down the slope and the y-axis perpendicular to the slope, find a solution to the Navier-Stokes equation for a flow of depth d down the slope under the assumptions that the flow is steady and uniform in the x-direction.

   If the flow rate was doubled, find the effect on the depth of the fluid, assuming that the flow remained uniform.

   If, instead of having a free surface at $y = d$, there was another solid boundary, what would the ratio of the new flow rate to the original flow rate be? You may assume that there is still no pressure gradient in the y direction.

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