The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. The surface displacement of low amplitude gravity waves on the surface of an inviscid, irrotational fluid where the effects of surface tension can be neglected is given by

$$y = \eta(x,t) = \epsilon\cos(kx - \omega t).$$

Find the potential of the flow assuming the fluid is of uniform depth, d, in the negative y-direction and that ϵ is small and so terms proportional to ϵ^2 and smaller can be ignored.

Find the relation between the frequency, ω, of the waves and their wavenumber, k.

Show that for short waves (k large) the effect of the bottom boundary becomes small.

Show that as the wavelength becomes large (k small) both the phase velocity and group velocities converge to the same limit which is to be determined.

Turn over ...

2. An inviscid fluid of constant density ρ moves with a steady velocity

$$\mathbf{u} = (Ax + By, Cx - Ay, 0),$$

where A, B and C are constants. Show that this flow satisfies the equation of continuity. If there is no external force, $\mathbf{F}$, show that this flow satisfies Euler's equation for a suitable pressure field which is to be found.

Find the relation between B and C if the flow is also irrotational. For this case (i) verify that the flow and the previously determined pressure satisfy Bernoulli's equation, and (ii) find the equations of the streamlines.

3. State the Milne-Thompson theorem for the complex potential of an inviscid flow past a circular cylinder. Show that the potential derived does have a streamline along the cylinder.

Show how the Milne-Thompson theorem can be used to derive the complex potential for the uniform flow past a circle of radius a centred at the origin with the flow in the far field parallel to the real axis.

If circulation Γ around the cylinder is added the complex potential becomes

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z.$$

Find the value of the circulation, Γ, such that there is only one stagnation point of the flow on the cylinder.

Use Blasius's theorem

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z$$

to find the force exerted on the cylinder for a general circulation.

Turn over ...

4. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   A flow towards a flat plate which lies along the x-axis gives rise to an exterior flow past the plate given by

   $$U(x) = Cx.$$

   By looking for a solution to the boundary layer equations in terms of the stream function ψ which takes the form

   $$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + ff'' - f'^2 + 1 = 0,$$

   where A, B, α and β are constants that are to be determined, verifying that the boundary layer is of constant width. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

5. A viscous fluid flows along the inside of a long circular cylinder of radius a whose axis lies along the z-axis. What are the boundary conditions for the fluid at the point of contact with the cylinder? Show that, subject to certain assumptions that are to be stated, the z-component of the velocity satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = \frac{G}{\rho\nu}$$

where G is the imposed pressure gradient in the z direction.

If the cylinder was lifted so that its axis were vertical, and the flow was driven by gravity and not by any imposed pressure gradient, how would the right hand side of the above equation be altered?

Find the flow under the effect of gravity in the cylinder and calculate the volume of water that would flow down the pipe every second.

A down-pipe from a $40\,\mathrm{m}^2$ roof must be able to cope with heavy rain falling at a rate of $36\,\mathrm{mm}$ in an hour. Calculate the minimum radius that the cylinder must have in order that the water would drain down the pipe according to your solution, assuming $\rho = 10^3\,\mathrm{kg\,m^{-3}}$, $g = 10\,\mathrm{m\,s^{-2}}$ and $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$. Comment on the validity of this solution.

[You may quote the result that in plane polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

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Internal Examiner:	Dr O.S. Kerr
External Examiners:	Professor S.A. Robertson
	Professor D.J. Needham