The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x,t) = \epsilon\cos(kx - \omega t).$$

Find the potential of the flow assuming the fluid is infinitely deep in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk,$$

where g is the acceleration due to gravity.

If surface tension is taken into consideration then the pressure of the fluid at the surface becomes

$$p = P_0 - T\frac{\partial^2\eta}{\partial x^2}$$

where T is a positive constant. Including the effect of gravity, show that the relation between the frequency and the wavenumber becomes

$$\omega^2 = gk + \frac{k^3 T}{\rho}.$$

Determine the phase velocity of these waves, and show that it has a minimum for some wavenumber, k, which is to be determined.

2. An inviscid fluid undergoing a 2-dimensional flow occupies the region $x \geq 0$. At $x = 0$ there is a plane rigid wall. A line vortex of strength Γ is located in the flow at $z = x_0 + iy_0$ with x_0 and y_0 real. Explain why the complex potential for the flow is

$$w(z) = -\frac{i\Gamma}{2\pi}\ln(z - x_0 - iy_0) + \frac{i\Gamma}{2\pi}\ln(z + x_0 - iy_0).$$

Show that the vortex moves parallel to the wall in the negative y-direction with speed $\Gamma/4\pi x_0$.

If $y_0 = 0$, show that the velocity of the fluid at the wall is given by

$$v = -\frac{\Gamma x_0}{\pi(y^2 + x_0^2)},$$

and show that at the wall $x = 0$,

$$\frac{\partial \phi}{\partial t} = -\frac{\Gamma^2}{4\pi^2(y^2 + x_0^2)}.$$

Hence, using Bernoulli's equation, show that the net force exerted on the wall by the vortex is

$$\int_{-\infty}^{\infty} p\,\mathrm{d}y = 0.$$

You may quote the results that

$$\int_{-\infty}^{\infty} \frac{1}{y^2 + a^2}\,dy = \frac{\pi}{a} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{(y^2 + a^2)^2}\,dy = \frac{\pi}{2a^3}.$$

3. A viscous fluid lies above the solid boundary $z = 0$. Its velocity is given by

$$\mathbf{u}(x, y, z) = \left(Axz, Ayz, Bz^2 + Cz + D\right).$$

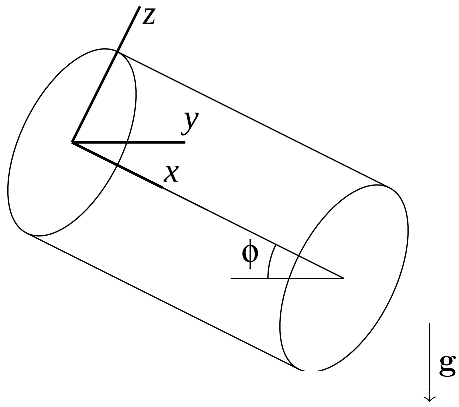
Determine B, C and D in terms of A so that this flow is both incompressible and satisfies the appropriate boundary condition at the plane $z = 0$. By considering its three components, show that this flow is an exact solution of the steady Navier-Stokes equation for some pressure which is to be determined.

Determine the vorticity of the flow, showing that vortex lines are circles parallel to the $z = 0$ plane and with centre on the z-axis.

Confirm that the ratio of the length of the vortex lines to the strength of the vorticity is constant. Why might this be surprising in this example?

Turn over ...

4. An incompressible viscous fluid flows down a long circular tube of radius a inclined at an angle ϕ to the horizontal. The absolute pressures at the ends of the tube are the same.



Using orthogonal coordinates with the x-axis aligned with the axis of the tube and the y-axis horizontal as indicated, find a solution for the flow along the tube under the assumptions that the flow is steady and uniform along the tube and the fluid fills the tube.

Calculate the pressure in the tube, verifying that surfaces of constant pressure are of the form $z = \text{constant}$.

Find the general formula for the total flow rate of fluid out of the pipe.

Another flow that can be derived from the above solution is the uniform flow down a tube half-filled with fluid, with the rest filled with air. What are the appropriate boundary conditions for the fluid at its surface? Why can this solution *not* be used to find flows down the tube which are exact solutions of the Navier-Stokes equation when the fluid fills less than half the tube?

[Hint: $\nabla^2(a^2 - y^2 - z^2) = -4$]

Turn over . . .

5. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

A plate lies along the positive x-axis. There is an exterior flow past the plate given by

$$U(x) = Cx^{2/3}.$$

By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

$$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

show that the problem can be reduced to the ordinary differential equation

$$f''' + ff'' + \frac{4}{5}\left(1 - f'^2\right) = 0,$$

where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

What are the boundary conditions satisfied by $f(\eta)$?

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