The Navier-Stokes equations for an incompressible Newtonian fluid are

$$\nabla\cdot\mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}.$$

Euler's equation for an incompressible inviscid fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}.$$

Bernoulli's equation for an incompressible inviscid irrotational fluid is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi = Q(t)$$

where $\mathbf{F} = -\nabla\Phi$ and $\mathbf{u} = \nabla\phi$.

You may neglect the effect of gravity in questions unless otherwise stated.

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1. Derive the boundary-layer equations for the steady two-dimensional incompressible flow of a viscous liquid along a plane impermeable surface. What is the appropriate form of the boundary conditions?

   A plate lies along the positive x-axis. There is an exterior flow past the plate given by

   $$U(x) = Cx^{1/2}.$$

   By looking for a solution to the boundary layer equations in terms of the streamfunction, ψ, which takes the form

   $$\psi = Ax^{\alpha} f(\eta), \quad \text{where} \quad \eta = \frac{By}{x^{\beta}},$$

   show that the problem can be reduced to the ordinary differential equation

   $$f''' + ff'' + \frac{2}{3}\left(1 - f'^2\right) = 0,$$

   where A, B, α and β are constants that are to be determined. *Do not try to solve this differential equation.*

   What are the boundary conditions satisfied by $f(\eta)$?

Turn over ...

2. State the Milne-Thompson theorem for the complex potential of a flow past a circle. Verify that the edge of the circle is a streamline of the flow.

   The complex potential for a stagnation point flow centred on the origin is

   $$w(z) = Az^2,$$

   where A is a constant. Find the corresponding potential if a cylinder of radius a is placed with its axis at the origin.

   If the circulation due to a vortex at the origin with potential

   $$-\frac{i\Gamma}{2\pi} \ln z$$

   is superimposed, write down the complex potential for the new flow. Using $z = re^{i\theta}$, find the real potential, $\phi(r, \theta)$, of this flow in terms of the polar coordinates r, θ.

   Find the circulation, Γ, for which there are exactly two stagnation points on the surface of the cylinder.

   [You may quote the result that the gradient of a function ϕ in polar coordinates (r, θ) is

   $$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}.$$]

3. The velocity of a viscous fluid is given by

   $$\mathbf{u}(x, y, z) = \Big(Bx, By, -A(x^2 + y^2) + Cz\Big).$$

   Determine C in terms of A and B to ensure that this flow is also incompressible.

   By considering its three components, show that this flow is an exact solution of the steady Navier-Stokes equation for some pressure which is to be determined.

   Determine the vorticity of the flow, showing that vortex lines are circles parallel to the $z = 0$ plane and with centre on the z-axis.

   Consider the flow in the plane $y = 0$. Show that $z = -Ax^2/(4B) + \alpha/x^2$, with α a constant, is the equation of the streamlines and sketch the flow.

Turn over ...

4. The surface of an inviscid irrotational fluid with gravity waves is given by

$$y = \eta(x,t) = \epsilon \cos(kx - \omega t)\,.$$

Find the potential of the flow assuming the fluid is of uniform depth h in the negative y-direction and that ϵ is small so terms proportional to ϵ^2 and smaller can be ignored.

Show that, if surface tension is neglected, the wavenumber, k, and frequency, ω, of the waves are related by

$$\omega^2 = gk \tanh kh,$$

where g is the acceleration due to gravity.

If surface tension is taken into consideration, the pressure of the fluid at the surface becomes

$$p = P_0 - T\frac{\partial^2 \eta}{\partial x^2}$$

where T is a positive constant. Show that the relation between the frequency and the wavenumber is then

$$\omega^2 = \left(g + \frac{k^2 T}{\rho}\right) k \tanh kh.$$

Show that for $T \neq 0$ the group velocity of the waves increases as the waves become shorter ($k \to \infty$), while for $T = 0$ it decreases.

Turn over . . .

5. From the Navier-Stokes equations derive the vorticity equation for an incompressible flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

A flow is given by

$$\mathbf{u}(x, y, z, t) = \left(u'(x, y, t),\ -Ay + v'(x, y, t),\ Az\right),$$

where A is constant. Show that the vorticity of this flow is of the form $\boldsymbol{\omega} = (0,\ 0,\ \omega_3(x, y, t))$ where ω_3 satisfies

$$\frac{\partial \omega_3}{\partial t} + u' \frac{\partial \omega_3}{\partial x} - Ay \frac{\partial \omega_3}{\partial y} + v' \frac{\partial \omega_3}{\partial y} = A\omega_3 + \nu \left(\frac{\partial^2 \omega_3}{\partial x^2} + \frac{\partial^2 \omega_3}{\partial y^2} \right).$$

Assume that the vorticity component ω_3, its derivatives, and the velocities u' and v' are small and so products of these quantities can be ignored. Show solutions to the resulting linear equation exist when the vorticity takes the form

$$\omega_3 = \epsilon f(y) e^{\lambda t} \cos kx,$$

with ϵ, λ and k constant, finding the ordinary differential equation that $f(y)$ must satisfy.

For the special case $\lambda + \nu k^2 = A$, show that the general solution to this differential equation is

$$f(y) = B \int e^{-(Ay^2/2\nu)}\, dy + C,$$

where B and C are constants.

[You may quote the result $\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u}$.]

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